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Proper invariant turbulence modelling within one-point statistics

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A new turbulence modelling approach is presented. Geometrically reformulating the averaged Navier–Stokes equations on a four-dimensional non-Riemannian manifold without changing the physical content of the theory, additional modelling restrictions which are absent in the usual Euclidean (3+1)-dimensional framework naturally emerge. The modelled equations show full form invariance for all Newtonian reference frames in that all involved quantities transform as true 4-tensors. Frame accelerations or inertial forces of any kind are universally described by the underlying four-dimensional geometry.

By constructing a nonlinear eddy viscosity model within the k–ϵ family for high turbulent Reynolds numbers the new invariant modelling approach demonstrates the essential advantages over current (3+1)-dimensional modelling techniques. In particular, new invariants are gained, which allow for a universal and consistent treatment of non-stationary effects within a turbulent flow. Furthermore, by consistently introducing via a Lie-group symmetry analysis a new internal modelling variable, the mean form-invariant pressure Hessian, it will be shown that already a quadratic nonlinearity is sufficient to capture secondary flow effects, for which in current nonlinear eddy viscosity models a higher nonlinearity is needed. In all, this paper develops a new unified formalism which will naturally guide the way in physical modelling whenever reasonings are based on the general concept of invariance.

1. Introduction

All turbulence models developed so far, irrespective of their statistical origin, averaged or filtered, have one common deficiency: their governing equations only show manifest ‘form invariance’ under space–time coordinate transformations when restricted to Galilei transformations but not to general coordinate transformations.

The notion of form invariance is clearly to be distinguished from that of ‘frame independence’. They are two distinct concepts which should not be confused whenever equations are transformed. After any variable transformation the former is defined as a property in which the structural form of an equation stays unchanged, while the latter is defined as a property in which the transformed equation is completely independent of all features that are being owned by the transformation. Demanding frame independence in an equation is thus a stronger condition than demanding form invariance, or equivalently, frame independence implies form invariance, but not conversely. The only exception occurs within Galilei transformations, since in

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classical physics they are the only transformation set to connect inertial frames of reference in which the two notions of form invariance and frame independence coincide. In this discussion it is important to note that in the physics community the concept of form invariance is frequently termed as covariance, whereas in the engineering community the concept of frame independence is frequently termed as objectivity.

For a more elaborate discussion on equational form invariance and frame independence within arbitrary variable transformations, see the recent work of Frewer (2009). In this paper, however, we are only concerned with the concept of form invariance relative to a special subset of variable transformations, i.e. to transformations in the 'coordinates' of time and space, also termed as passive transformations (Sexl & Urbantke 2001).

The principle objective of this paper is to introduce a qualitatively new mathematical framework in which turbulence modelling can be performed in a very general sense and to point out the essential differences between this new framework and the one used so far. Its development will show that it is the appropriate and even the ultimate framework whenever reasonings in classical turbulence modelling are based on invariance. The development itself will be carried out in the Reynolds-averaged Navier–Stokes (RANS) context of ensemble averages; the translation to subgrid-scale modelling in the large-eddy simulation (LES) context can then be done straightforwardly. After all, the new mathematical framework is fully decoupled from a certain statistical averaging technique.

The basic idea behind this framework is of geometrical nature. With the methods of differential geometry it is possible to resolve the above-mentioned deficiency which all turbulent models are currently sharing. The aim is to allow for modelling procedures within equations which show manifest form invariance under 'arbitrary' space–time coordinate transformations, with the only restriction that no relativistic physics is induced, such that, after such a transformation, all dynamical quantities still continue to evolve in the sense of Newtonian mechanics. This results in an alternative mathematical representation for the theory of classical fluid mechanics, in that the geometrical embedding is changed without modifying the physical content of the theory.

The usual three-dimensional space manifold has to be replaced by a true four-dimensional space–time manifold, which, conceptually as well as mathematically, will be the 'classical' limit of the manifold used in Einstein's general theory of relativity (Einstein 1916). In the following, the former manifold will be symbolized as the (3+1)-dimensional manifold to indicate that the additional time coordinate only acts as an evolution parameter for any physical object in that manifold, whereas the latter manifold will be denoted as the four-dimensional manifold to indicate that the time coordinate now behaves as an independent variable next to the spatial variables.

By construction this new four-dimensional manifold is to be classified as a Newtonian manifold, since unlike in Einsteinian physics, space and time measurements are uncorrelated in it. In particular the connection between inertial frames of references is still carried by the Galilei transformations. Embedding a classical theory into a Newtonian four-dimensional manifold is thus not restricted to fluid mechanics only; it can be applied to the whole field of Newtonian physics, as to thermodynamics or electrostatics. Only the full time-dependent theory of electrodynamics has to be to excluded, since all electromagnetic phenomena are of a relativistic nature.

Now, what is the real advantage of using a different geometrical representation for a given theory? The answer surely depends on what one intends to do. If any equation need to be solved, analytically or numerically, a reformulation from a

(3+1)-dimensional to a four-dimensional setting will be of no advantage at all, but if the equation is not closed and needs to be modelled, as a material law or as any equation of turbulence, such reformulations automatically bring along consistent and structured modelling arguments in the most natural way whenever they are based on invariant principles.

The claim here is that '(3+1)-dimensional modelling is not equivalent to four-dimensional modelling' within Newtonian physics. In other words, only within a four-dimensional space–time manifold invariant classical turbulence modelling can be performed properly. Its clear superiority over the usual (3+1)-dimensional approach can be fixed by the following arguments which will be discussed in detail in the upcoming sections:

(*a*) The variables of space and time are fully independent.

This implies that in any closure strategy not only space but also time derivatives have to be considered, hence allowing not only for a universal and consistent treatment of curvature effects but also for a universal and consistent treatment of non-stationary effects.

(*b*) Physical quantities as velocities or stresses always transform as tensors, irrespective of whether they are objective (frame independent) or not.

This is important when modelling unclosed quantities as for example the Reynolds-stress tensor with non-objective quantities.

(*c*) Frame accelerations or inertial forces of any kind can be interpreted as a pure geometrical effect, described by the affine connection of the four-dimensional manifold.

This implies that inertial and non-inertial turbulence do not need to be modelled separately anymore. A four-dimensional turbulence model will describe non-inertial turbulence as rotation, swirling or curved surfaces equally well or equally bad as the corresponding inertial case.

(*d*) The special space–time structure of the four-dimensional manifold allows for additional modelling constraints, which are not present in the usual (3+1)-dimensional geometrical formulation.

The paper is organized as follows: § 2 will construct the necessary space–time manifold for turbulence modelling as a classical limit from relativistic physics. This limit leads to a non-Riemannian manifold having two subspaces in which a unique but singular metrical connection can be defined and to a set of coordinate transformations in space and time, which are compatible with this manifold. Section 3 then develops the general algorithm for geometrically reformulating any given physical equation on a flat four-dimensional Newtonian manifold, which will be done explicitly for the incompressible, isothermal Navier–Stokes equations, and § 4 for the corresponding ensemble-averaged Navier–Stokes equations. Section 5 exemplifies this new formalism by preparing an algebraic closure for the one-point, first-moment averaged Navier–Stokes equations. Subsection 5.1 then proposes for high turbulent Reynolds numbers a qualitatively new nonlinear eddy viscosity model (EVM) of the k–ϵ family showing a universal structure for all Newtonian reference frames. Section 6 finally discusses this new proposition.

2. Construction of the classical Newtonian space–time manifold

In the subsequent development we consider a four-dimensional smooth manifold in which each point can be smoothly labelled by four coordinates x^α. A manifold is characterized by its geometrical structure in that it is either endowed with an affine connection or additionally endowed with a metric. For a concise definition of these geometrical concepts, see Appendix A. Our aim is to construct a physical manifold $\mathcal{N}$

(a) in which the four coordinates x^α are identified as three spatial coordinates x^i and a time coordinate x^0 (Greek indices will always run from $0, \dots, 3$, while Latin indices will only run from $1, \dots, 3$),

(b) which possesses the minimal amount of geometrical complexity and

(c) in which physics evolves on the basis of a Newtonian description emerging as a classical limit from Einstein's theory of relativity.

In other words, our aim is to do Newtonian physics in a true four-dimensional formulation with the minimal possible complexity.

To fulfil the requirement of geometrical simplicity our manifold $\mathcal{N}$ should be fixed such that it is always possible to globally choose a coordinate system in which the affine connection vanishes, defining it as our 'standard coordinate system'. This implies that the manifold is flat, that it is globally without any curvature – physically the feature of global flatness is the approximative result of allowing only for small mass scales, i.e. allowing the gravitational force to decouple from the space–time geometry, which certainly is a valid limit for all technical mechanics. Choosing geometrical representations with a higher degree of complexity will not lead to any new insights regarding turbulence modelling within technological flows.

The only difference between Newtonian mechanics and Einsteinian mechanics lies in the single postulate of a constant speed of light for all local inertial reference frames. The two remaining postulates, general covariance and the equivalence between inertial and gravitational mass, are not characteristic features of Einsteinian mechanics. Newtonian mechanics can always be mathematically reformulated such that it conforms with these two principles, but then, Newtonian mechanics has to generally evolve in a 'curved' non-Riemannian space–time manifold as was pioneered by Cartan (1923) and Friedrichs (1928) and then further developed by various authors such as Havas (1964), Trautman (1966) and Ehlers (1991): this non-relativistic theory of gravitation in a curved space–time manifold, known as the Newton–Cartan theory, reveals the close resemblance between Einstein's and Newton's theories in that gravitation in both cases is identified as a geometrical feature.

It is interesting to note that not to regard the principle of general covariance as an own exclusive postulate of general relativity, which states that all laws of physics have to be of the same structural form in all space–time coordinate systems, was already recognized by Kretschmann as early as 1917 (Kretschmann 1917). On this point we will return in more detail later on.

Since Einsteinian mechanics assumes that there exists a maximum signal velocity equal to the speed of light c in an inertial system, while Newtonian mechanics assumes that there exist signals propagating with infinite velocity so that points in space are causally connected even if no time goes by, the whole concept of Newtonian mechanics on a space–time manifold will emerge from Einsteinian mechanics simply in the classical limit $c \rightarrow \infty$. Given that our manifold is flat the limit can be taken in the Minkowskian manifold of special relativity with the pseudo-Euclidean metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{2.1}$$

if the four-dimensional coordinate system is to be chosen as a global inertial system with Cartesian coordinates (standard coordinate system). The metric can be used to define an invariant infinitesimal line element which has either the dimension of length

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}\hat{x}^\mu\,\mathrm{d}\hat{x}^\nu, \text{ with } \hat{x}^\mu = (ct, x^i) \tag{2.2}$$

or the dimension of time

$$\mathrm{d}\tau^2 = \hat{\eta}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \text{ with } \hat{\eta}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\dfrac{1}{c^2}\mathbb{1} \end{pmatrix} \text{ and } x^\mu = (t, x^i). \tag{2.3}$$

To perform the classical limit $c \to \infty$ we will use the latter one. Together with its inverse metric $\hat{\eta}^{\mu\nu}$ the relationship

$$\hat{\eta}^{\alpha\lambda}\hat{\eta}_{\lambda\beta} = \delta^\alpha_\beta, \text{ with } \hat{\eta}^{\alpha\lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -c^2\mathbb{1} \end{pmatrix}, \tag{2.4}$$

which equivalently can be written as

$$\check{\eta}^{\alpha\lambda}\hat{\eta}_{\lambda\beta} = -\frac{1}{c^2}\delta^\alpha_\beta, \text{ with } \check{\eta}^{\alpha\lambda} = \begin{pmatrix} -\dfrac{1}{c^2} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{2.5}$$

degenerates to the following relation as $c \to \infty$:

$$h^{\alpha\lambda}g_{\lambda\beta} = 0, \text{ with } h^{\alpha\lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \text{ and } g_{\lambda\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.6}$$

where the metrics $\check{\eta}^{\alpha\beta}$ and $\hat{\eta}_{\alpha\beta}$ become independent and equal to the tensors $h^{\alpha\beta}$ and $g_{\alpha\beta}$ respectively. In other words, in the limit $c \to \infty$ the unique non-singular Minkowski metric $\eta_{\mu\nu}$ splits up into two in space and time separate singular tensors, which can be identified as a space-like metric $h^{\alpha\beta}$ and a time-like metric $g_{\alpha\beta}$. Thus our so constructed classical Newtonian space–time manifold $\mathscr{N}$ is a non-Riemannian manifold; its geometrical structure shows, in contrast to a Riemannian manifold, a non-unique and singular metrical connection.

The next step is to determine the coordinate transformations that are compatible with this manifold $\mathscr{N}$. Taking the limit $c \to \infty$ in the relation for the invariant infinitesimal line element of (2.3)

$$\mathrm{d}\tau^2 = \hat{\eta}_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \underset{c\to\infty}{=} g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \mathrm{d}t^2 \tag{2.7}$$

leads to the statement that the time coordinate $x^0 = t$ itself is an invariant. Thus the space–time coordinate transformations compatible with the classical Newtonian manifold $\mathscr{N}$ are those in which the time coordinate, up to an additive constant, transforms as an absolute quantity

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta), \text{ with } \tilde{x}^0 = x^0. \tag{2.8}$$

Since the coordinate differentials $\mathrm{d}\tilde{x}^\alpha$ transform as a contravariant 4-vector and since the differential time coordinate transforms invariantly $\mathrm{d}\tilde{x}^0 = \mathrm{d}x^0$, one can construct a new fundamental contravariant 4-vector

$$\frac{\mathrm{d}\tilde{x}^\alpha}{\mathrm{d}\tilde{x}^0} = \frac{\partial\tilde{x}^\alpha}{\partial x^\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}x^0} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \tilde{u}^\alpha = \frac{\partial\tilde{x}^\alpha}{\partial x^\beta}u^\beta, \text{ with } u^\alpha = (1, u^i); \tag{2.9}$$

the velocity vector u^α is to be identified as a pure time-like contravariant vector, since its time component always transforms numerical invariantly as $\tilde{u}^0 = u^0 = 1$ and thus can never vanish. Regarding fluid mechanics the definition (2.9) is to be seen as the transition going from the Lagrangian to the Eulerian description in which the velocity vector then turns into a velocity vector field $u^\alpha = u^\alpha(x^\beta)$.

From the velocity vector field one can now construct another fundamental contravariant kinematic quantity, the acceleration vector field

$$\frac{\mathrm{d}\tilde{u}^\alpha}{\mathrm{d}\tilde{x}^0} = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}x^0} \quad \Longleftrightarrow \quad \tilde{a}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} a^\beta, \text{ with } a^\alpha = (0, a^i), \tag{2.10}$$

which is to be identified as a pure space-like contravariant vector. Thus within a true four-dimensional Newtonian manifold $\mathscr{N}$ velocity fields will always evolve differently than accelerations or forces, in the sense that velocity fields u^α will always evolve as time-like vectors, while accelerations a^α or forces, according to Newton's second law $F^\alpha \sim a^\alpha$, will always evolve or act as space-like vectors. This kinematical distinction between certain physical fields, which is definitely absent in the usual (3+1)-dimensional formulation, will serve as a new powerful restriction property when closing any equation of turbulence.

Now, since $\mathscr{N}$ is a non-Riemannian manifold it is impossible to assign to 'every' contravariant 4-vector a corresponding covariant 4-vector, and conversely. One rather has to distinguish between two classes of 4-vectors, the class of space-like contravariant vectors $a^\alpha = (0, a^i)$ and the class of time-like covariant vectors $w_\alpha = (w_0, 0)$, where in each class one can 'uniquely' assign a corresponding covariant or contravariant vector respectively. Following the notation of Havas (1964) the tensors necessary to generate this correspondence in each class, which were derived again in more detail and were completed in Frewer (2009), show an analogous singular space–time relation as in (2.6) but with non-constant component entries

$$k_{\alpha\lambda} m^{\lambda\beta} = 0, \text{ with } k_{\alpha\lambda} = \begin{pmatrix} \|\boldsymbol{u}\|^2 & -u^j \\ -u^i & \mathbb{1} \end{pmatrix} \text{ and } m^{\lambda\beta} = \begin{pmatrix} 1 & u^j \\ u^i & u^i u^j \end{pmatrix}, \tag{2.11}$$

where $\boldsymbol{u} = (u^1, u^2, u^3)$ are the spatial components of the 4-velocity field $u^\alpha = (1, \boldsymbol{u})$ and the squared spatial norm defined as $\|\boldsymbol{u}\|^2 = k_{ij} u^i u^j$. The singular symmetric tensor $k_{\alpha\beta}$ is to be interpreted as the corresponding covariant space-like metric of $h^{\alpha\beta}$ in the class of space-like contravariant vectors a^α, while the singular symmetric tensor $m^{\alpha\beta}$ as the corresponding contravariant time-like metric of $g_{\alpha\beta}$ in the class of time-like covariant vectors w_α. This correspondence is not restricted to 4-vectors only but applies to all tensors $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ of any rank if only each component can be identified either as space-like or as time-like. For all other classes of tensors there is no unambiguous correspondence between covariance and contravariance.

Eventually our non-Riemannian manifold $\mathscr{N}$ allows for four different tensors, which in well-defined subspaces of $\mathscr{N}$ behave as metrical tensors to measure distances and/or to measure time, showing the expected result that in Newtonian physics, unlike in Einsteinian physics, space and time measurements are uncorrelated.

Up to now we characterized all metric tensors only in the standard coordinate system (Cartesian and inertial). In arbitrary coordinate systems they are characterized by their invariant property of a vanishing covariant derivative but only in those subspaces in which their metrical property applies:

$$\nabla_\lambda h^{ij} = 0, \quad \nabla_\lambda k_{ij} = 0; \quad \nabla_\lambda g_{00} = 0, \quad \nabla_\lambda m^{00} = 0. \tag{2.12}$$

The affine connection, however, is zero in the standard coordinate system, which is always possible to implement as a reference frame due to the flatness of the manifold $\mathscr{N}$. Then 'relative' to this standard frame the affine connection in arbitrary coordinate systems is given according to its non-tensorial transformation property (A 7) simply as

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \frac{\partial \tilde{x}^{\rho}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}. \tag{2.13}$$

3. General covariance of the Navier–Stokes equations

When Einstein formulated his general theory of relativity (Einstein 1916), he was proud to present for the first time a theory that was generally form invariant or, as he first called it, a theory that was generally covariant; its equations retained their structural form under 'arbitrary' transformations of the space–time coordinate system.

Only a year later the mathematician Kretschmann (1917) objected: the property of general covariance is no peculiarity of the new gravitational theory presented by Einstein. He argued that any space–time theory can be formulated in a generally covariant form; a so-called principle of general covariance would even be fully devoid of any physical content. In the literature this statement became famous as Kretschmann's objection. For further discussions on it, Einstein's response and the still-active debate that followed, see the papers of Norton (1993, 1995) and Dieks (2006).

Kretschmann's objection that general covariance is physically vacuous, in that it does not limit or restrict the range of acceptable theories, is a rather 'non-trivial' objection when it comes down to constructing and formulating 'new' physical theories, as it was at that time in 1917 for the theory of general relativity. However, for already-existing physical theories his objection is more or less obvious from a pure theoretical point of view, since at the end, only a new mathematical representation is given for the theory. But this does not mean that it is always an easy task to put any given theory into a generally covariant form; sometimes it's a challenge to the mathematician's ingenuity.

Since our aim is only to achieve general covariance of Newtonian mechanics in a flat manifold $\mathscr{N}$, the procedure is simple and defined as follows.

(i) Write the Newtonian equations in the inertial (3+1)-dimensional Cartesian form.

(ii) Rewrite them into the corresponding four-dimensional form (standard coordinate system) using the geometrical structure of the Newtonian space–time manifold $\mathscr{N}$.

(iii) Make the transition from inertial Cartesian to arbitrary space–time coordinates by replacing the partial derivative with the covariant derivative $\partial_\alpha \to \nabla_\alpha$.

For the Navier–Stokes equations of an incompressible, isothermal fluid the programme in each step reads

$$\left.\begin{aligned} \partial_i u^i &= 0, \\ \partial_t u^i + u^j \partial_j u^i &= -\delta^{ij} \partial_j p + \nu \delta^{jk} \partial_j \partial_k u^i, \end{aligned}\right\} \tag{3.1}$$

$$\downarrow$$

$$\left.\begin{aligned} \partial_\alpha u^\alpha &= 0, \\ u^\beta \partial_\beta u^\alpha &= -h^{\alpha\beta} \partial_\beta p + \nu h^{\beta\gamma} \partial_\beta \partial_\gamma u^\alpha, \end{aligned}\right\} \tag{3.2}$$

$$\downarrow$$

$$\left.\begin{aligned} \nabla_\alpha u^\alpha &= 0, \\ u^\beta \nabla_\beta u^\alpha &= -h^{\alpha\beta} \nabla_\beta p + \nu h^{\beta\gamma} \nabla_\beta \nabla_\gamma u^\alpha. \end{aligned}\right\} \tag{3.3}$$

By construction, the Navier–Stokes equations in the last formulation will stay form invariant under all space–time coordinate transformations which are compatible with the underlying four-dimensional Newtonian manifold $\mathscr{N}$. In other words, if we change the frame of reference according to (2.8) the Navier–Stokes equations in form (3.3) will transform form invariantly:

$$\left.\begin{aligned} \tilde{\nabla}_\alpha \tilde{u}^\alpha &= 0, \\ \tilde{u}^\beta \tilde{\nabla}_\beta \tilde{u}^\alpha &= -\tilde{h}^{\alpha\beta}\tilde{\nabla}_\beta \tilde{p} + \nu \tilde{h}^{\beta\gamma}\tilde{\nabla}_\beta \tilde{\nabla}_\gamma \tilde{u}^\alpha. \end{aligned}\right\} \tag{3.4}$$

As was already discussed briefly in § 1, the property of form invariance does not imply the property of frame independence. Irrespective of being in a form-invariant representation, system (3.3) will in general show frame dependence emerging partly from the space-like metric $\tilde{h}^{\alpha\beta}$ (Frewer 2009) and partly from the affine connection in the covariant differentiation

$$\tilde{\nabla}_\alpha \tilde{u}^\beta = \tilde{\partial}_\alpha \tilde{u}^\beta + \tilde{u}^\lambda \tilde{\Gamma}^\beta_{\alpha\lambda}. \tag{3.5}$$

Furthermore, it is clear that the above reformulation did not change the physical content of the classical Navier–Stokes equations. System (3.3) will lead to the very same physical predictions as the original system (3.1) would do. In the reasoning of modelling turbulent flows, however, the two systems will lead to different results, as we now want to demonstrate in the remaining sections.

4. General covariance of the averaged Navier–Stokes equations

To show the essential results in the covariant development of the averaged Navier–Stokes equations it is fully sufficient to explicitly demonstrate it only relative to one-point statistics of ensemble averages up to the first moment. The development for higher statistical moments, or for N-point statistics, or even for a different choice in the statistical method itself can be done straightforwardly without any theoretical complications.

Using the Reynolds decomposition to separate the average and the fluctuating parts in the velocity and pressure fields with vanishing average fluctuations

$$u^\alpha = \langle u^\alpha \rangle + u'^\alpha, \quad p = \langle p \rangle + p', \quad \text{with } \langle u'^\alpha \rangle = 0, \ \langle p' \rangle = 0, \tag{4.1}$$

by defining the average as an ensemble average

$$\langle u^\alpha \rangle := \frac{1}{N}\sum_{r=1}^{N} (u^\alpha)^{(r)}, \quad \langle p \rangle := \frac{1}{N}\sum_{r=1}^{N} p^{(r)}, \tag{4.2}$$

where $N \gg 1$ is the number of realizations, it implies that

$$\langle u^\alpha \rangle = (1, \langle u^i \rangle), \quad u'^\alpha = (0, u'^i), \tag{4.3}$$

the average 4-velocity $\langle u^\alpha \rangle$, like the instantaneous velocity u^α, is a pure time-like vector which can never turn space-like and that the fluctuating 4-velocity u'^α is a pure space-like vector which can never turn time-like. The four-dimensional formulation thus shows that the average and fluctuating 4-velocities evolve differently within the Newtonian space–time manifold $\mathscr{N}$. The behaviour of the fluctuating velocity is not that of a velocity but rather that of an acceleration or that of a force, an information which is completely absent in the usual (3+1)-dimensional formulation.

Inserting decomposition (4.1) into the form-invariant Navier–Stokes equations (3.3) and then averaging these equations will straightforwardly lead to the general manifest form-invariant averaged Navier–Stokes equations

$$
\begin{gathered}
\nabla_\alpha \langle u^\alpha \rangle = 0, \\
\langle u^\beta \rangle \nabla_\beta \langle u^\alpha \rangle = -h^{\alpha\beta} \nabla_\beta \langle p \rangle + \nu h^{\beta\gamma} \nabla_\beta \nabla_\gamma \langle u^\alpha \rangle - \nabla_\beta \tau^{\alpha\beta},
\end{gathered}
\tag{4.4}
$$

where the Reynolds-stress tensor $\tau^{\alpha\beta} = \langle u'^\alpha u'^\beta \rangle$ is a pure 'space-like' 4-tensor, which has to be respected during modelling. The relevant averaged subspaces, in which a unique but singular metrical connection can be defined, provide, when written in the standard coordinate system (Cartesian and inertial), the two space-like metrics

$$
h^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \overset{\langle u \rangle}{k}_{\alpha\beta} = \begin{pmatrix} \| \langle \boldsymbol{u} \rangle \|^2 & -\langle u^j \rangle \\ -\langle u^i \rangle & \mathbb{1} \end{pmatrix} \tag{4.5}
$$

and the two time-like metrics

$$
g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \underset{\langle u \rangle}{m}^{\alpha\beta} = \begin{pmatrix} 1 & \langle u^j \rangle \\ \langle u^i \rangle & \langle u^i \rangle \langle u^j \rangle \end{pmatrix}. \tag{4.6}
$$

The averaged and fluctuating 4-velocities can then be uniquely represented as

$$
\langle u^\alpha \rangle = \underset{\langle u \rangle}{m}^{\alpha\beta} \langle u_\beta \rangle, \quad u'^\alpha = h^{\alpha\beta} u'_\beta, \tag{4.7}
$$

together with their inverse relations

$$
\langle u_\alpha \rangle = g_{\alpha\beta} \langle u^\beta \rangle, \quad u'_\alpha = \overset{\langle u \rangle}{k}_{\alpha\beta} u'^\beta, \tag{4.8}
$$

which define the covariant averaged and the covariant fluctuating 4-velocity respectively.

5. Invariant modelling of a nonlinear EVM

To exemplify the main advantages of modelling on a true four-dimensional manifold it is already sufficient to consider only the simplest closure strategy; i.e. in the following we will only focus on a local algebraic closure of the Reynolds-stress tensor. Within this limitation it is reasonable to consider the construction of a complete model. For an easy *ad hoc* comparison with already existing nonlinear turbulent EVMs it is worthwhile to construct a two-equation model which will be part of the k–ϵ family.

The aim is to close the Reynolds-stress tensor algebraically,

$$
\tau^{\alpha\beta} = F^{\alpha\beta}(\mathscr{V}), \tag{5.1}
$$

in which $\mathscr{V}$ represents the general closure set, here being defined as a local functional argument set of averaged variables. Before we begin to fix this set, it is advisable to first list the modelling restrictions for the unknown functional $F^{\alpha\beta}$:

(i) it is a contravariant tensor of rank 2,
(ii) it is a pure space-like tensor,
(iii) it is a symmetric tensor,
(iv) it carries the dimension of velocity squared.

Hence, just by geometrically reformulating the averaged Navier–Stokes equations on a four-dimensional manifold two more restrictions, namely (i) and (ii), are gained than in the usual (3+1)-dimensional formulation, where only the properties of symmetry (iii) and dimensional consistency (iv) serve as restrictions. The distinction between space-like and time-like as well as between contravariant and covariant tensor components in 'all' possible reference frames is completely absent in the (3+1)-dimensional formulation. In a non-Riemannian four-dimensional formulation, however, such a distinction is already of relevance in the standard reference frame, as it represents an immediate consequence of the underlying geometrical structure.

If the motivation in closing the averaged Navier–Stokes equations (4.4) is only to make use of the sole information these equations can supply, the most basic ansatz for a local algebraic closure is to close the Reynolds-stress tensor (5.1) with those variables which are being solved for, together with their first gradients:

$$\mathscr{V} = \{\langle P^\sigma\rangle, \langle u^\sigma\rangle; \nabla_\lambda\langle P^\sigma\rangle, \nabla_\lambda\langle u^\sigma\rangle\}, \tag{5.2}$$

where $\langle P^\alpha\rangle := h^{\alpha\beta}\nabla_\beta\langle p\rangle$ is the mean space-like pressure gradient. Of course this ansatz is not restricted in only taking the first gradients along; it can include gradients of any order – there is no upper limit for that. The only reason to truncate the functional set $\mathscr{V}$ at first order is to keep the degree of complexity to a necessary minimum.

Compared to current nonlinear EVMs developed by Speziale (1987), Gatski & Speziale (1993), Shih, Zhu & Lumley (1995), Craft, Launder & Suga (1996), Apsley & Leschziner (1998) and Wallin & Johansson (2000), which all can be traced back to the general series expansion of the Reynolds-stress tensor in terms of strain and vorticity tensors as first given by Pope (1975), the key differences in the structure and in the choice of the above closure set (5.2) are as follows:

(*a*) all functional variables in $\mathscr{V}$ transform as four-dimensional tensors under 'arbitrary' coordinate transformations with absolute time;

(*b*) on a four-dimensional manifold not only space derivatives but also time derivatives are automatically included;

(*c*) the mean pressure gradient is taken along; and

(*d*) not only the mean four-dimensional velocity gradient, which can be split up into a symmetric mean strain and an antisymmetric mean vorticity tensor, but also the 4-velocity itself are part of the closure set $\mathscr{V}$.

As will be discussed in more detail later on, such a modelling approach will induce not only the correct and consistent treatment of curvature effects in a turbulent flow but also the correct and consistent treatment of non-local and memory effects in that flow. It is clear that only the first two statements are a consequence of four-dimensional modelling, while the last two statements have to be seen as a new algebraic modelling proposal which could have also been carried out in (3+1)-dimensional modelling – but here with the decisive advantage that it is carried out systematically within the range of universal form invariance.

The necessity of including the mean pressure gradient as an additional independent closure variable can only be justified upon looking at flow configurations in which current nonlinear EVMs still have serious difficulty in resolving a turbulent state properly (Speziale 1991; Pope 2000; Leschziner 2001). Despite their success as to returning the correct level of anisotropy or capturing the influence of curvature in any turbulent flow, there are still some basic flow configurations for which the modelling hypothesis of a pure relationship between stress and mean velocity gradient completely fails, irrespective of the degree of nonlinearity. To this class certainly belong all flows in which turbulence is subjected to a mean deformation on a finite length or time scale, either by a sudden change in the bounding geometry or by a change in an acting body force. For example, regarding the components of the Reynolds-stress tensor all current nonlinear EVMs cannot properly account for the relaxation effects that will emerge after deformation; only the models of the next level of closure, the second-moment closure models, show the ability to describe these effects.

Even worse, 'after' the deformation of an initially homogeneous turbulent flow, known as the return-to-isotropy problem, any stress model that exhibits a sole dependence on the mean velocity gradient will inevitably predict that the Reynolds-stress anisotropies are zero due to the non-existence of a mean velocity in that domain. This prediction of vanishing anisotropies is in severe conflict with the data of experiments and simulations (Choi & Lumley 2001), which show instead that the anisotropies generated 'during' deformation even show a long persistence 'after' deformation with a slow decay rate relative to a turbulent time scale. For flow configurations with a rapidly changing bounding geometry the mean pressure gradient would be an optimal candidate to account for non-local effects, which surely are necessary to understand the dynamics behind such flows – in the case of the return-to-isotropy problem, however, such that beyond the deformation domain no averaged velocity profile is induced.

A more singular problem which current nonlinear EVMs face is that all wall-bounded flows which show a symmetric averaged velocity profile, as in ducts or channels, have an extremal centre position which corresponds to the annulation of the mean velocity gradient, which again implies vanishing Reynolds-stress anisotropies at that position. But this is certainly not the case; the Reynolds-stress anisotropies show clear finite values at that position, which corresponds to a 'local' non-validity of the current nonlinear viscosity modelling hypothesis in that the Reynolds-stress tensor should globally depend on velocity gradients only. For this singular problem it is not out of the question that pressure can play again the role of a *deus ex machina* variable. As is well known from turbulent channel flow, the averaged pressure shows a simple linear profile in the direction of the mean flow but a rather complex profile in the wall-normal direction being proportional to the wall-normal Reynolds-stress component (Pope 2000).

After all, there is no deeper reason why the Reynolds-stress tensor should not depend on the mean pressure gradient, as long as this dependence is not in conflict with any physical constraints — on this point we will return later on.

As is well known, the averaged Navier–Stokes equations (4.4) with variable choice (5.2) as a closed system are incapable of describing any turbulent flow; it surely needs information from outside to fix certain scales of turbulence. Since the present focus is only on incompressible, isothermal flows the minimum requirement for a complete description is to fix at least two scales, a turbulent length scale l_T and a turbulent time scale t_T, where each scale need to be determined dynamically for every space–time point. Using the Kolmogorov phenomenology of turbulent flows (Kolmogorov 1941*a*,*b*; Davidson 2004), the two scales l_T and t_T can be equivalently replace by the two functionally independent quantities of a turbulent kinetic energy $\mathscr{K}$ and its dissipation rate ϵ.

On a four-dimensional manifold the turbulent kinetic energy is given as the invariant $\mathscr{K} = \langle u'^{\alpha} u'_{\alpha}\rangle/2$, while the dissipation rate of turbulent kinetic energy has the invariant structure

$$\epsilon = 2\nu k^{\langle u\rangle}_{\alpha\beta} k^{\langle u\rangle}_{\mu\nu} \langle s^{\alpha\mu} s^{\nu\beta}\rangle, \text{ with } s^{\alpha\beta} = \tfrac{1}{2}(h^{\alpha\lambda}\nabla_\lambda u'^{\beta} + h^{\beta\lambda}\nabla_\lambda u'^{\alpha}) \tag{5.3}$$

as the contravariant fluctuating 4-strain-rate tensor. Usually the true dissipation rate of turbulent kinetic energy ϵ is not taken, but rather the pseudo-dissipation rate (Pope 2000)

$$\mathscr{E} = \nu h^{\mu\lambda} \langle \nabla_\mu u'^{\alpha} \nabla_\lambda u'_{\alpha}\rangle, \tag{5.4}$$

which is related to the true dissipation rate by $\mathscr{E} = \epsilon - \nu\nabla_\alpha\nabla_\beta \tau^{\alpha\beta}$.

Hence, the complete closure variable list for a four-dimensional invariant two-equation turbulence model up to gradients of first order is given as

$$\mathscr{V} = \{\mathscr{K}, \mathscr{E}, \langle P^\sigma \rangle, \langle u^\sigma \rangle; \nabla_\lambda \mathscr{K}, \nabla_\lambda \mathscr{E}, \nabla_\lambda \langle P^\sigma \rangle, \nabla_\lambda \langle u^\sigma \rangle\}, \tag{5.5}$$

where for consistency also the first-order gradients of $\mathscr{K}$ and $\mathscr{E}$ have to be included. The invariant turbulent kinetic energy $\mathscr{K}$ evolves according to the transport equation

$$\langle u^\alpha \rangle \nabla_\alpha \mathscr{K} = -k^{\langle u \rangle}_{\beta\lambda} \tau^{\lambda\alpha} \nabla_\alpha \langle u^\beta \rangle - \mathscr{E} + \nabla_\lambda \mathscr{D}^\lambda_{(\mathscr{K})} + \nu h^{\alpha\beta} \nabla_\alpha \nabla_\beta \mathscr{K}, \tag{5.6}$$

and its invariant (pseudo-)dissipation rate $\mathscr{E}$ according to

$$\begin{aligned}\langle u^\alpha \rangle \nabla_\alpha \mathscr{E} = {} & \mathscr{P}^{\alpha}_{(1)\,\beta} \nabla_\alpha \langle u^\beta \rangle + \mathscr{P}^{\alpha\beta}_{(2)\,\lambda} \nabla_\alpha \nabla_\beta \langle u^\lambda \rangle + \mathscr{P}_{(3)} \\ & - \Upsilon + \nabla_\lambda \mathscr{D}^\lambda_{(\mathscr{E})} + \nu h^{\alpha\beta} \nabla_\alpha \nabla_\beta \mathscr{E}. \end{aligned} \tag{5.7}$$

These complete form-invariant transport equations can be easily derived from the usual (3+1)-dimensional equations for $\mathscr{K}$ and $\mathscr{E}$ just by making use of the simple algorithm as shown for the Navier–Stokes equations (3.1)–(3.3).

Next to the Reynolds-stress tensor $\tau^{\alpha\beta}$ and its modelling restrictions mentioned in the beginning of this section the following unclosed four-dimensional invariant turbulent flow quantities have to be modelled:

(*a*) $\mathscr{P}^{\alpha}_{(1)\,\beta} = -2\nu(h^{\lambda\sigma} \langle \nabla_\lambda u'_\beta \nabla_\sigma u'^\alpha \rangle + h^{\lambda\alpha} \langle \nabla_\lambda u'^\sigma \nabla_\beta u'_\sigma \rangle)$, as a dissipation production tensor of mixed rank (1,1) with the properties that

(i) the contravariant component α is space-like, which allows for the unique representation $\mathscr{P}^{\alpha}_{(1)\,\beta} = h^{\alpha\lambda} \mathscr{P}_{(1)\lambda\beta}$;

(ii) the covariant component β is time-like but contributes in the $\mathscr{E}$-equation only for $\beta \neq 0$, since $\nabla_\alpha \langle u^0 \rangle = 0$ in all Newtonian reference frames;

(iii) it carries the space-like dimension $l_T^2/t_T^3 = \mathscr{E}$.

(*b*) $\mathscr{P}^{\alpha\beta}_{(2)\,\lambda} = -2\nu h^{\alpha\sigma} \langle u'^\beta \nabla_\sigma u'_\lambda \rangle$, as a dissipation production tensor of mixed rank (2,1) with the properties that

(i) the contravariant components α and β are space-like, which allows for the unique representation $\mathscr{P}^{\alpha\beta}_{(2)\,\lambda} = h^{\alpha\rho} h^{\beta\sigma} \mathscr{P}_{(2)\rho\sigma\lambda}$;

(ii) the covariant component λ is time-like but contributes in the $\mathscr{E}$-equation only for $\lambda \neq 0$, since $\nabla_\alpha \nabla_\beta \langle u^0 \rangle = 0$ in all Newtonian reference frames;

(iii) it carries the space-like dimension $l_T^3/t_T^3 = \mathscr{K}^{3/2}$.

(*c*) $\mathscr{P}_{(3)} = -2\nu h^{\alpha\beta} \langle \nabla_\alpha u'^\lambda \nabla_\beta u'^\sigma \nabla_\sigma u'_\lambda \rangle$, as a dissipation production scalar which carries the dimension $l_T^2/t_T^4 = \mathscr{E}^2/\mathscr{K}$.

(*d*) $\Upsilon = 2\nu^2 h^{\alpha\beta} h^{\rho\sigma} \langle \nabla_\alpha \nabla_\rho u'^\lambda \nabla_\beta \nabla_\sigma u'_\lambda \rangle$, as a dissipation destruction scalar carrying the dimension $l_T^2/t_T^4 = \mathscr{E}^2/\mathscr{K}$.

(*e*) $\mathscr{D}^\lambda_{(\mathscr{E})} = -\nu h^{\alpha\sigma}(\langle u'^\lambda \nabla_\alpha u'^\rho \nabla_\sigma u'_\rho \rangle + 2\langle \nabla_\alpha u'^\lambda \nabla_\sigma p' \rangle)$, as a dissipation diffusion tensor of rank (1,0), which is space-like and with dimension $l_T^3/t_T^4 = \mathscr{K}^{1/2}\mathscr{E}$.

(*f*) $\mathscr{D}^\lambda_{(\mathscr{K})} = -\langle u'^\alpha u'_\alpha u'^\lambda \rangle/2 - \langle p' u'^\lambda \rangle$, as a turbulent kinetic energy diffusion tensor of rank (1,0), which is space-like and carries the dimension $l_T^3/t_T^3 = K^{3/2}$.

Due to its numerous unclosed terms, the $\mathscr{E}$-equation certainly causes the most difficulty in modelling turbulent flows, especially wall-bounded flows. To capture the complete domain of such flows it is necessary to split up each unclosed term in the $\mathscr{E}$-equation into at least two contributions, one showing an 'explicit' dependence on the molecular viscosity parameter ν and the other an 'implicit' dependence which will be carried by the dissipation rate $\mathscr{E}$ as a dimensional scale. Based on the local turbulent Reynolds number

$$Re_T = \frac{\mathscr{K}^2}{\nu\mathscr{E}} =: \frac{\nu_T}{\nu}, \tag{5.8}$$

which for any given ν or, equivalently, for any given global Reynolds number Re is always small (low) in the near-wall region and large (high) around the centre region of the flow, each unclosed tensor $X^{\alpha\beta\cdots}_{\rho\sigma\ldots}$ in the $\mathscr{E}$-equation will thus be split up into a low-Reynolds-number part $\check{X}^{\alpha\beta\cdots}_{\rho\sigma\ldots}$ and into a high-Reynolds-number part $\hat{X}^{\alpha\beta\cdots}_{\rho\sigma\ldots}$ accordingly:

$$\left.\begin{aligned}
\mathscr{P}^{\alpha}_{(1)\,\beta} &= \nu\check{\mathscr{P}}^{\alpha}_{(1)\,\beta} + \hat{\mathscr{P}}^{\alpha}_{(1)\,\beta}, \text{ with the local dimensional structure } \left[\mathscr{P}^{\alpha}_{(1)\,\beta}\right] = \nu\frac{\mathscr{E}^2}{\mathscr{K}^2} + \mathscr{E},\\
\mathscr{P}^{\alpha\beta}_{(2)\,\lambda} &= \nu\check{\mathscr{P}}^{\alpha\beta}_{(2)\,\lambda} + \hat{\mathscr{P}}^{\alpha\beta}_{(2)\,\lambda}, \text{ with } \left[\mathscr{P}^{\alpha\beta}_{(2)\,\lambda}\right] = \nu\frac{\mathscr{E}}{\mathscr{K}^{1/2}} + \mathscr{K}^{3/2},\\
\mathscr{P}_{(3)} &= \nu\check{\mathscr{P}}_{(3)} + \hat{\mathscr{P}}_{(3)}, \text{ with } \left[\mathscr{P}_{(3)}\right] = \nu\frac{\mathscr{E}^3}{\mathscr{K}^3} + \frac{\mathscr{E}^2}{\mathscr{K}},\\
\Upsilon &= \nu^2\check{\Upsilon} + \hat{\Upsilon}, \text{ with } [\Upsilon] = \nu^2\frac{\mathscr{E}^4}{\mathscr{K}^5} + \frac{\mathscr{E}^2}{\mathscr{K}},\\
\mathscr{D}^{\lambda}_{(\mathscr{E})} &= \nu\check{\mathscr{D}}^{\lambda}_{(\mathscr{E})} + \hat{\mathscr{D}}^{\lambda}_{(\mathscr{E})}, \text{ with } \left[\mathscr{D}^{\lambda}_{(\mathscr{E})}\right] = \nu\frac{\mathscr{E}^2}{\mathscr{K}^{3/2}} + \mathscr{K}^{1/2}\mathscr{E}.
\end{aligned}\right\} \tag{5.9}$$

The low-Re_T and high-Re_T terms are modelled in those turbulent flow regions in which they become relevant. Close to the wall, where $\mathscr{K} \sim 0$ and $\mathscr{E} \sim$ constant, simple dominant balance arguments in the dimensional analysis given above reveal that in this limit the full $\mathscr{E}$-equation (5.7) will turn into a low-Re_T transport equation

$$\begin{aligned}
\langle u^\alpha\rangle\nabla_\alpha\mathscr{E} = {}& \nu\check{\mathscr{P}}^{\alpha}_{(1)\,\beta}\nabla_\alpha\langle u^\beta\rangle + \nu\check{\mathscr{P}}^{\alpha\beta}_{(2)\,\lambda}\nabla_\alpha\nabla_\beta\langle u^\lambda\rangle + \nu\check{\mathscr{P}}_{(3)}\\
&- \nu^2\check{\Upsilon} + \nu\nabla_\lambda\check{\mathscr{D}}^{\lambda}_{(\mathscr{E})} + \nu h^{\alpha\beta}\nabla_\alpha\nabla_\beta\mathscr{E}.
\end{aligned} \tag{5.10}$$

The high-Re_T terms are modelled towards the opposite extreme within a double limit: (i) far away from any solid walls, where $\mathscr{K}$ is non-vanishing and where gradients of any averaged flow quantities are rather weak, and (ii) in turbulent flows with a high global Reynolds number Re or equivalently in the limit of $\nu \to 0$, where $\mathscr{K} \gg 1$ and where in fully developed turbulence $\mathscr{E}$ nearly attains a finite constant value. In this sense the full $\mathscr{E}$-equation (5.7) will turn into a high-Re_T transport equation

$$\langle u^\alpha\rangle\nabla_\alpha\mathscr{E} = \hat{\mathscr{P}}^{\alpha}_{(1)\,\beta}\nabla_\alpha\langle u^\beta\rangle + \hat{\mathscr{P}}^{\alpha\beta}_{(2)\,\lambda}\nabla_\alpha\nabla_\beta\langle u^\lambda\rangle - \hat{\Psi} + \nabla_\lambda\hat{\mathscr{D}}^{\lambda}_{(\mathscr{E})}, \tag{5.11}$$

where the scalar production and destruction rate have been merged into one invariant rate $\hat{\Psi} = -\hat{\mathscr{P}}_{(3)} + \hat{\Upsilon}$, since it is impossible to distinguish between these two quantities during any invariant modelling process.

The aim of this paper is really to demonstrate the mechanisms of modelling turbulence on a four-dimensional manifold, to give the spirit of it. In this sense it is fully sufficient not to demonstrate it within the complete domain of a turbulent flow but only within the range of the high-Re_T transport equation (5.11).

5.1. *Construction of a high-Re_T turbulence model*

Dropping the irrelevant molecular diffusion terms in the averaged momentum and turbulent kinetic energy transport equations in the limit of high Re_T, the full set of coupled model equations for the averaged velocity and pressure field reads

$$\left.\begin{aligned} \nabla_\alpha \langle u^\alpha \rangle &= 0, \\ \langle u^\beta \rangle \nabla_\beta \langle u^\alpha \rangle &= -h^{\alpha\beta} \nabla_\beta \langle p \rangle - \nabla_\beta \tau^{\alpha\beta}, \\ \langle u^\alpha \rangle \nabla_\alpha \mathscr{K} &= -k^{\langle u \rangle}_{\beta\lambda} \tau^{\lambda\alpha} \nabla_\alpha \langle u^\beta \rangle - \mathscr{E} + \nabla_\lambda \mathscr{D}^\lambda_{(\mathscr{K})}, \\ \langle u^\alpha \rangle \nabla_\alpha \mathscr{E} &= \hat{\mathscr{P}}^{\alpha}_{(1)\beta} \nabla_\alpha \langle u^\beta \rangle + \hat{\mathscr{P}}^{\alpha\beta}_{(2)\lambda} \nabla_\alpha \nabla_\beta \langle u^\lambda \rangle - \hat{\Psi} + \nabla_\lambda \hat{\mathscr{D}}^\lambda_{(\mathscr{E})}. \end{aligned}\right\} \tag{5.12}$$

All six unclosed tensor functions

$$\tau^{\alpha\beta}, \quad \mathscr{D}^\lambda_{(\mathscr{K})}, \quad \hat{\mathscr{D}}^\lambda_{(\mathscr{E})}, \quad \hat{\mathscr{P}}^{\alpha}_{(1)\beta}, \quad \hat{\mathscr{P}}^{\alpha\beta}_{(2)\lambda}, \quad \hat{\Psi} \tag{5.13}$$

will be modelled on the basis of the same functional dependency structure $\mathscr{V}$ as given in (5.5). However, a prior Lie-group analysis (Olver 1993; Ibragimov 1994; Bluman & Kumei 1996; Cantwell 2002), which is beyond the scope of this paper, clearly reveals that in order to model consistently along 'all' Lie-point symmetries of the Navier–Stokes equation, the set $\mathscr{V}$ has to be fixed as

$$\mathscr{V} = \left\{ \frac{1}{\mathscr{K}^{1/2}} \langle u^\sigma \rangle ; \frac{\mathscr{K}^{1/2}}{\mathscr{E}} \nabla_\lambda \mathscr{K}, \frac{\mathscr{K}^{3/2}}{\mathscr{E}^2} \nabla_\lambda \mathscr{E}, \frac{\mathscr{K}^2}{\mathscr{E}^2} \nabla_\lambda \langle P^\sigma \rangle, \frac{\mathscr{K}}{\mathscr{E}} \nabla_\lambda \langle u^\sigma \rangle \right\}, \tag{5.14}$$

in which the pressure gradient of first order $\langle P^\sigma \rangle$ has to be excluded and in which only variables with non-dimensionalized space-like components are to be used. It can be briefly explained as follows: in the standard coordinate system (Cartesian and inertial) the complete infinite-dimensional Lie-point algebra of the 'full' Navier–Stokes equations $\mathscr{L}_{NS}$, that means, irrespective of whether the flow field is turbulent or laminar, is spanned by the following seven linear independent generators:

$$\left.\begin{aligned} \mathsf{T} = \partial_t, \quad \mathsf{S} &= 2t\partial_t + x^i \partial_i - u^i \partial_{u^i} - 2p\partial_p, \\ \mathsf{R}_{ij} &= x^i \partial_j - x^j \partial_i + u^i \partial_{u^j} - u^j \partial_{u^i}, \; i \neq j, \\ \mathsf{G}(\boldsymbol{f}(t)) &= f^i(t)\partial_i + f'^i(t)\partial_{u^i} - f''^i(t) x^i \partial_p, \quad \mathsf{P}(g(t)) = g(t)\partial_p, \end{aligned}\right\} \tag{5.15}$$

where $\boldsymbol{f}$ and g are arbitrary time-dependent differentiable functions. For the corresponding RANS system of equations (5.12) the Lie-point symmetries equivalently translate to (Ünal 1997; Oberlack 2001)

$$\left.\begin{aligned} \mathsf{T} = \partial_t, \quad \mathsf{S} &= 2t\partial_t + x^i \partial_i - \langle u^i \rangle \partial_{\langle u^i \rangle} - 2\langle p \rangle \partial_{\langle p \rangle} - 2\mathscr{K}\partial_{\mathscr{K}} - 4\mathscr{E}\partial_{\mathscr{E}}, \\ \mathsf{R}_{ij} &= x^i \partial_j - x^j \partial_i + \langle u^i \rangle \partial_{\langle u^j \rangle} - \langle u^j \rangle \partial_{\langle u^i \rangle}, \; i \neq j \\ \mathsf{G}(\boldsymbol{f}(t)) &= f^i(t)\partial_i + f'^i(t)\partial_{\langle u^i \rangle} - f''^i(t) x^i \partial_{\langle p \rangle}, \quad \mathsf{P}(g(t)) = g(t)\partial_{\langle p \rangle}, \end{aligned}\right\} \tag{5.16}$$

where each symmetry transformation then has to be seen as a modelling restriction for the unclosed system (5.12). For obtaining the symmetry generators in arbitrary reference frames they, of course, have to be transformed accordingly: however, such a coordinate transformation will not change the number or the physical properties of the symmetries; only the mathematical representation is changed which in general can turn arbitrarily complex.

Since the autonomous property of the Navier–Stokes equations is not changed during modelling, the time translation symmetry generated by T will always be trivially fulfilled. The same holds true for the fixed rotation symmetry generated by R_{ij}, since as being itself a coordinate transformation connecting inertial frames, modelling occurs throughout in a tensorial formulation. The scaling symmetry generated by S can be reduced to a scaling symmetry of the physical dimensions of length and time. Thus even if the modelling procedure respects dimensional consistency, then also this scaling symmetry will always be automatically fulfilled. On the other hand, the pure time-dependent pressure translation symmetry generated by P is in general only fulfilled if the modelling procedure allows for pressure variables in which at least one spatial derivative is involved. Certainly, this will always be the case if the mean space-like pressure gradient $\langle P^{\alpha}\rangle$, or one of its higher order gradients, is used as a modelling variable. However, the extended Galilei symmetry generated by G, which allows for arbitrary linear frame accelerations if only the pressure field adjusts the induced inertial force accordingly, demands in general the exclusion of the first-order mean pressure gradient $\langle P^{\alpha}\rangle$ as a modelling variable, since its pressure correction term emerging from such a transformation cannot always be compensated by the remaining transformed modelling variables chosen herein.

Altogether, this briefly explains result (5.14). However, it must be clear that this result, although being expressed in a four-dimensional covariant form, is not a consequence due to the four-dimensional formulation itself. It also could have been gained within the usual (3+1)-dimensional formulation. The reason is that since the physical content of the Navier–Stokes equations remains unchanged under the transition given here of going from a Euclidean (3+1)-dimensional towards a flat non-Riemannian four-dimensional geometry, a Lie-group symmetry analysis will inevitably lead to the very same results for each framework. In other words, from the perspective of a Lie-group symmetry analysis the only difference between the (3+1)-dimensional and the four-dimensional Navier–Stokes equations is that the latter equations are already inherently carrying all reference frame representations. But, as already said, a Lie-point symmetry will not change its physical and/or mathematical interpretations under 'any' coordinate transformation. A coordinate transformation will only change the mathematical representation of the symmetry but not its meaning and its consequences which it can further imply (Ovsiannikov 1982; Olver 1993).

Now, when orientating the upcoming modelling procedure towards current nonlinear EVMs, which in general provide a broad range of applicability, it is reasonable to choose for each unclosed tensor function the following dependencies:

$$\left.\begin{aligned}
\tau^{\alpha\beta} &= \tau^{\alpha\beta}(\mathscr{V}_{\tau}), \text{ with } \mathscr{V}_{\tau} = \left\{\tfrac{1}{\mathscr{K}^{1/2}}\langle u^{\sigma}\rangle;\ \tfrac{\mathscr{K}^{2}}{\mathscr{E}^{2}}\nabla_{\lambda}\langle P^{\sigma}\rangle,\ \tfrac{\mathscr{K}}{\mathscr{E}}\nabla_{\lambda}\langle u^{\sigma}\rangle\right\} \subset \mathscr{V},\\
\mathscr{D}^{\lambda}_{(\mathscr{K})} &= \mathscr{D}^{\lambda}_{(\mathscr{K})}(\mathscr{V}_{\mathscr{D}_{\mathscr{K}}}), \text{ with } \mathscr{V}_{\mathscr{D}_{\mathscr{K}}} = \mathscr{V}_{\tau} \cup \left\{\tfrac{\mathscr{K}^{1/2}}{\mathscr{E}}\nabla_{\lambda}\mathscr{K}\right\} \subset \mathscr{V},\\
\hat{\mathscr{D}}^{\lambda}_{(\mathscr{E})} &= \hat{\mathscr{D}}^{\lambda}_{(\mathscr{E})}(\mathscr{V}_{\hat{\mathscr{D}}_{\mathscr{E}}}), \text{ with } \mathscr{V}_{\hat{\mathscr{D}}_{\mathscr{E}}} = \mathscr{V}_{\tau} \cup \left\{\tfrac{\mathscr{K}^{3/2}}{\mathscr{E}^{2}}\nabla_{\lambda}\mathscr{E}\right\} \subset \mathscr{V},\\
\hat{\mathscr{P}}^{\alpha}_{(1)\,\beta} &= \hat{\mathscr{P}}^{\alpha}_{(1)\,\beta}(\mathscr{V}_{\hat{\mathscr{P}}_{1}}), \text{ with } \mathscr{V}_{\hat{\mathscr{P}}_{1}} = \mathscr{V}_{\tau},\\
\hat{\mathscr{P}}^{\alpha\beta}_{(2)\,\lambda} &= \hat{\mathscr{P}}^{\alpha\beta}_{(2)\,\lambda}(\mathscr{V}_{\hat{\mathscr{P}}_{2}}), \text{ with } \mathscr{V}_{\hat{\mathscr{P}}_{2}} = \mathscr{V}_{\tau},\\
\hat{\Psi} &= \hat{\Psi}(\mathscr{V}_{\hat{\Psi}}), \text{ with } \mathscr{V}_{\hat{\Psi}} = \mathscr{V}.
\end{aligned}\right\} \tag{5.17}$$

For example the dependence choice for the two production terms of the $\mathscr{E}$-equation is orientated along the production term in the $\mathscr{K}$-equation (5.12) which is proportional to the Reynolds-stress tensor, whereas the dependence choice for the two diffusion terms will explicitly account for a diffusion source term in each of the two turbulent-scale transport equations. On the other hand, the dependency range of the unclosed scalar term is not restricted; it is chosen to be modelled via the full dependence (5.14).

The algebraic theory of tensor invariants, which was prepared for classical continuum mechanics by Spencer & Rivlin (1958), shows that the most general polynomial expansion of any tensor function $\mathscr{F}^{\alpha\beta\cdots}_{\rho\sigma\ldots}(\mathscr{X})$ naturally truncates at a fixed order $N_{\mathscr{F}}$. In other words, the number of independent invariants and linearly independent tensors that may be formed from the elements of any given tensor set $\mathscr{X}$ is finite. This means that the general expansion of $\mathscr{F}^{\alpha\beta\cdots}_{\rho\sigma\ldots}(\mathscr{X})$ can be expressed as a finite polynomial in which the expansion coefficients are functions of a finite number of invariants based on the elements of $\mathscr{X}$.

In continuum mechanics the assumption of material frame indifference can be frequently made for certain materials as for solids or ordinary dense fluids in order to reduce possible constitutive equations. However, as indicated already by Lumley (1970), despite some analogies between the dynamics of a turbulent flow and the behaviour of a nonlinear viscoelastic fluid, an assumption as that of material frame indifference is unfounded. Various experiments and direct numerical simulations clearly support this statement in that unclosed terms of turbulence, as the Reynolds-stress tensor, always show strong frame dependency. Hence, a principle of material frame indifference may not be used for turbulence modelling, not even in an approximative sense – for a more elaborate discussion on this topic, see Frewer (2009).

The first lowest order to model explicit frame dependence in the unclosed tensor functions (5.17) is to truncate the functional expansions at quadratic order. However, when demanding an overall quadratic nonlinearity in the transport equations (5.12), as we will do herein, the two dissipation production tensors have to be truncated already at linear order, while the remaining four unclosed terms have to be truncated at quadratic order.

The invariants belonging to the tensor set $\mathscr{V}$ (5.14) up to quadratic order are given by the following 12 dimensionless expressions:

$$\left.\begin{array}{c}
\mathscr{I}_1 = h^{\alpha\beta}k^{\langle u\rangle}_{\alpha\beta}, \quad \mathscr{I}_2 = \dfrac{\mathscr{K}^2}{\mathscr{E}^2}\nabla_\alpha\langle P^\alpha\rangle, \quad \mathscr{I}_3 = \dfrac{1}{\mathscr{E}}\langle u^\alpha\rangle\nabla_\alpha\mathscr{K}, \quad \mathscr{I}_4 = \dfrac{\mathscr{K}}{\mathscr{E}^2}\langle u^\alpha\rangle\nabla_\alpha\mathscr{E}, \\
\mathscr{I}_5 = \dfrac{\mathscr{K}}{\mathscr{E}^2}h^{\alpha\beta}\nabla_\alpha\mathscr{K}\cdot\nabla_\beta\mathscr{K}, \quad \mathscr{I}_6 = \dfrac{\mathscr{K}^2}{\mathscr{E}^3}h^{\alpha\beta}\nabla_\alpha\mathscr{K}\cdot\nabla_\beta\mathscr{E}, \quad \mathscr{I}_7 = \dfrac{\mathscr{K}^3}{\mathscr{E}^4}h^{\alpha\beta}\nabla_\alpha\mathscr{E}\cdot\nabla_\beta\mathscr{E}, \\
\mathscr{I}_8 = \dfrac{\mathscr{K}^4}{\mathscr{E}^4}\nabla_\alpha\langle P^\alpha\rangle\cdot\nabla_\beta\langle P^\beta\rangle, \; \mathscr{I}_9 = \dfrac{\mathscr{K}^4}{\mathscr{E}^4}\nabla_\alpha\langle P^\beta\rangle\cdot\nabla_\beta\langle P^\alpha\rangle, \; \mathscr{I}_{10} = \dfrac{\mathscr{K}^3}{\mathscr{E}^3}\nabla_\alpha\langle P^\beta\rangle\cdot\nabla_\beta\langle u^\alpha\rangle, \\
\mathscr{I}_{11} = \dfrac{\mathscr{K}^2}{\mathscr{E}^2}\nabla_\alpha\langle u^\beta\rangle\cdot\nabla_\beta\langle u^\alpha\rangle, \quad \mathscr{I}_{12} = \dfrac{\mathscr{K}^2}{\mathscr{E}^2}h^{\lambda\sigma}k^{\langle u\rangle}_{\alpha\beta}\nabla_\lambda\langle u^\alpha\rangle\cdot\nabla_\sigma\langle u^\beta\rangle.
\end{array}\right\} \tag{5.18}$$

It is worth noting that only the invariants $\mathscr{I}_3$ and $\mathscr{I}_4$ carry a time derivative in all Newtonian reference frames; all others are pure space-like invariants. In particular, these two invariants represent the kinematic left-hand sides of the $\mathscr{K}$- and $\mathscr{E}$-transport equations respectively – mathematically we are thus confronted with a self-consisting modelling approach which can account for turbulent memory effects. On the contrary, $\mathscr{I}_1$ is a numerical invariant which always has the constant value $\mathscr{I}_1 = 3$ in all Newtonian reference frames.

Respecting the modelling constraints for the Reynolds-stress tensor $\tau^{\alpha\beta}$ given in the beginning of § 5, as well as the consistency condition that its contraction has to be twice the turbulent kinetic energy $k^{\langle u\rangle}_{\alpha\beta}\tau^{\alpha\beta} = 2\mathscr{K}$, a quadratic expansion of $\tau^{\alpha\beta}(\mathscr{V}_\tau)$ (5.17) will then be of the form

$$\tau^{\alpha\beta} = \tfrac{2}{3}h^{\alpha\beta}\mathscr{K} + a^{\alpha\beta}, \text{ with } a^{\alpha\beta} = \mathscr{K}\sum_{i=1}^{8}\Lambda_{(i)}a^{\alpha\beta}_{(i)}, \tag{5.19}$$

where $a^{\alpha\beta}$ is the space-like anisotropy tensor satisfying $k^{\langle u\rangle}_{\alpha\beta}a^{\alpha\beta} = 0$. The expansion coefficients will be chosen to be functions of the full invariant set $\mathscr{V}$ in order to achieve the maximum range of applicability when restricting the model physically: $\Lambda_{(i)} = \Lambda_{(i)}(\mathscr{I}_{(j)})$, with $1 \leqslant j \leqslant 12$. The anisotropic dimensionless expansion terms are

$$\left.\begin{aligned}
a^{\alpha\beta}_{(1)} &= \frac{\mathscr{K}^2}{\mathscr{E}^2}\Big(h^{\alpha\rho}h^{\beta\sigma} - \tfrac{1}{3}h^{\alpha\beta}h^{\rho\sigma}\Big)\nabla_\rho\nabla_\sigma\langle p\rangle, \quad a^{\alpha\beta}_{(2)} = \frac{\mathscr{K}}{\mathscr{E}}\Big(h^{\alpha\lambda}\nabla_\lambda\langle u^\beta\rangle + h^{\beta\lambda}\nabla_\lambda\langle u^\alpha\rangle\Big),\\
a^{\alpha\beta}_{(3)} &= \frac{\mathscr{K}^4}{\mathscr{E}^4}h^{\kappa\lambda}\Big(h^{\alpha\rho}h^{\beta\sigma} - \tfrac{1}{3}h^{\alpha\beta}h^{\rho\sigma}\Big)\nabla_\rho\nabla_\sigma\langle p\rangle\cdot\nabla_\kappa\nabla_\lambda\langle p\rangle,\\
a^{\alpha\beta}_{(4)} &= \frac{\mathscr{K}^4}{\mathscr{E}^4}h^{\sigma\lambda}\Big(h^{\alpha\rho}h^{\beta\kappa} - \tfrac{1}{3}h^{\alpha\beta}h^{\rho\kappa}\Big)\nabla_\rho\nabla_\sigma\langle p\rangle\cdot\nabla_\kappa\nabla_\lambda\langle p\rangle,\\
a^{\alpha\beta}_{(5)} &= \frac{\mathscr{K}^3}{\mathscr{E}^3}\Big(h^{\alpha\rho}h^{\beta\sigma} - \tfrac{2}{3}h^{\alpha\beta}h^{\rho\sigma} + h^{\beta\rho}h^{\alpha\sigma}\Big)\nabla_\rho\nabla_\lambda\langle p\rangle\cdot\nabla_\sigma\langle u^\lambda\rangle,\\
a^{\alpha\beta}_{(6)} &= \frac{\mathscr{K}^2}{\mathscr{E}^2}\Big(h^{\alpha\rho}\nabla_\rho\langle u^\sigma\rangle\cdot\nabla_\sigma\langle u^\beta\rangle - \tfrac{2}{3}h^{\alpha\beta}\nabla_\rho\langle u^\sigma\rangle\cdot\nabla_\sigma\langle u^\rho\rangle + h^{\beta\rho}\nabla_\rho\langle u^\sigma\rangle\cdot\nabla_\sigma\langle u^\alpha\rangle\Big),\\
a^{\alpha\beta}_{(7)} &= \frac{\mathscr{K}^2}{\mathscr{E}^2}h^{\rho\sigma}\Big(\nabla_\rho\langle u^\alpha\rangle\cdot\nabla_\sigma\langle u^\beta\rangle - \tfrac{1}{3}h^{\alpha\beta}k^{\langle u\rangle}_{\kappa\lambda}\nabla_\rho\langle u^\kappa\rangle\cdot\nabla_\sigma\langle u^\lambda\rangle\Big),\\
a^{\alpha\beta}_{(8)} &= \frac{\mathscr{K}^2}{\mathscr{E}^2}k^{\langle u\rangle}_{\kappa\lambda}\Big(h^{\alpha\rho}h^{\beta\sigma} - \tfrac{1}{3}h^{\alpha\beta}h^{\rho\sigma}\Big)\nabla_\rho\langle u^\kappa\rangle\cdot\nabla_\sigma\langle u^\lambda\rangle,
\end{aligned}\right\} \tag{5.20}$$

where the notation for the mean pressure gradient has been written out, $\langle P^\alpha\rangle = h^{\alpha\beta}\nabla_\beta\langle p\rangle$, to explicitly show the required index symmetry in the expansion. When looking carefully at the above expansion one observes that the mean velocity field variable $\langle u^\alpha\rangle$ is excluded, although it is part of the dependency set $\mathscr{V}_\tau$ (5.17). The expansion of the Reynolds-stress tensor $\tau^{\alpha\beta}$ only includes its gradient $\nabla_\alpha\langle u^\beta\rangle$. However, this would no longer be the case if, for example, the dependency set $\mathscr{V}_\tau$ is extended to the full set $\mathscr{V}$ (5.14) followed by a polynomial expansion in those variables up to a cubic nonlinearity. Then additional anisotropic terms like $a^{\alpha\beta}_{(1)}\mathscr{I}_3 \sim a^{\alpha\beta}_{(1)}\langle u^\alpha\rangle\nabla_\alpha\mathscr{K}$, or like $a^{\alpha\beta}_{(2)}\mathscr{I}_4 \sim a^{\alpha\beta}_{(2)}\langle u^\alpha\rangle\nabla_\alpha\mathscr{E}$, will enter the Reynolds-stress tensor. In any case, if the velocity field enters it can only enter in a contracted form, since its ‘time-like’ behaviour will always naturally exclude any uncontracted forms in the ‘space-like’ Reynolds-stress tensor.

Since expansion (5.20) only shows a quadratic nonlinearity, one can claim, following the results of Craft *et al.* (1996) and Speziale, Younis & Berger (2000), that the above-proposed algebraic model for the Reynolds-stress tensor will be incapable over a range of flows to correctly describe any effect of streamline curvature or, more specifically, will fail to predict the emergence and the subsequent persistence of a swirling flow component in fully developed axially rotating pipe flows, since it needs at least a cubic nonlinearity to capture all these effects. The combined approach of modelling on a four-dimensional manifold and to include the mean pressure as a new internal modelling variable will most naturally allow to capture these effects already on the level of a quadratic nonlinearity in the Reynolds-stress tensor, if proposed as in (5.20). It is essentially the anisotropy term $a^{\alpha\beta}_{(5)}$, the coupling between mean pressure and mean velocity, which is responsible for this. For the axially rotating pipe it is demonstrated in Appendix B that this term, then proportional to the radial pressure gradient, will give rise to a persisting swirl velocity. This definitely sheds new light on nonlinear EVMs as to identify the mechanisms responsible for generating such secondary flows, since currently (Speziale *et al.* 2000) it is claimed that only the mean axial velocity within a cubic nonlinearity can give rise to a swirling flow component in axially rotating pipes. Similar arguments will hold true for capturing the effects of more general flow curvatures in turbulence.

The quadratic expansion of the turbulent kinetic diffusion vector $\mathscr{D}^{\lambda}_{(\mathscr{K})}(\mathscr{V}_{\mathscr{D}_{\mathscr{K}}})$ (5.17) can only take the following structure if one respects its restriction of being a space-like vector:

$$\mathscr{D}^{\lambda}_{(\mathscr{K})} = \mathscr{K}^{3/2} \sum_{i=1}^{7} \Theta_{(i)} b^{\lambda}_{(i)}, \tag{5.21}$$

with the expansion coefficients $\Theta_{(i)} = \Theta_{(i)}(\mathscr{I}_{(j)})$ again as functions of the full invariant set (5.18) and the dimensionless expansion terms as

$$\left.\begin{aligned} b^{\lambda}_{(1)} &= \frac{\mathscr{K}^{1/2}}{\mathscr{E}} h^{\lambda\sigma}\nabla_{\sigma}\mathscr{K}, \\ b^{\lambda}_{(2)} &= \frac{\mathscr{K}^{3/2}}{\mathscr{E}^2}\langle u^{\alpha}\rangle\nabla_{\alpha}\langle P^{\lambda}\rangle, \quad b^{\lambda}_{(3)} = \frac{\mathscr{K}^{1/2}}{\mathscr{E}}\langle u^{\alpha}\rangle\nabla_{\alpha}\langle u^{\lambda}\rangle, \\ b^{\lambda}_{(4)} &= \frac{\mathscr{K}^{5/2}}{\mathscr{E}^3} h^{\lambda\rho}\nabla_{\rho}\mathscr{K}\cdot\nabla_{\sigma}\langle P^{\sigma}\rangle, \quad b^{\lambda}_{(5)} = \frac{\mathscr{K}^{5/2}}{\mathscr{E}^3} h^{\lambda\rho}\nabla_{\sigma}\mathscr{K}\cdot\nabla_{\rho}\langle P^{\sigma}\rangle, \\ b^{\lambda}_{(6)} &= \frac{\mathscr{K}^{3/2}}{\mathscr{E}^2} h^{\lambda\rho}\nabla_{\sigma}\mathscr{K}\cdot\nabla_{\rho}\langle u^{\sigma}\rangle, \quad b^{\lambda}_{(7)} = \frac{\mathscr{K}^{3/2}}{\mathscr{E}^2} h^{\rho\sigma}\nabla_{\rho}\mathscr{K}\cdot\nabla_{\sigma}\langle u^{\lambda}\rangle. \end{aligned}\right\} \tag{5.22}$$

The expansion structure of the high-Re_T dissipation diffusion vector $\hat{\mathscr{D}}^{\lambda}_{(\mathscr{E})}\big(\mathscr{V}_{\hat{\mathscr{D}}_{\mathscr{E}}}\big)$ (5.17) is of course up to dimensional considerations similar:

$$\hat{\mathscr{D}}^{\lambda}_{(\mathscr{E})} = \mathscr{K}^{1/2}\mathscr{E} \sum_{i=1}^{7} \hat{\Phi}_{(i)} \hat{c}^{\lambda}_{(i)}, \tag{5.23}$$

with $\hat{\Phi}_{(i)} = \hat{\Phi}_{(i)}(\mathscr{I}_{(j)})$ and

$$\left.\begin{aligned} \hat{c}^{\lambda}_{(1)} &= \frac{\mathscr{K}^{3/2}}{\mathscr{E}^2} h^{\lambda\sigma}\nabla_{\sigma}\mathscr{E}, \\ \hat{c}^{\lambda}_{(2)} &= \frac{\mathscr{K}^{3/2}}{\mathscr{E}^2}\langle u^{\alpha}\rangle\nabla_{\alpha}\langle P^{\lambda}\rangle, \quad \hat{c}^{\lambda}_{(3)} = \frac{\mathscr{K}^{1/2}}{\mathscr{E}}\langle u^{\alpha}\rangle\nabla_{\alpha}\langle u^{\lambda}\rangle, \\ \hat{c}^{\lambda}_{(4)} &= \frac{\mathscr{K}^{7/2}}{\mathscr{E}^4} h^{\lambda\rho}\nabla_{\rho}\mathscr{E}\cdot\nabla_{\sigma}\langle P^{\sigma}\rangle, \quad \hat{c}^{\lambda}_{(5)} = \frac{\mathscr{K}^{7/2}}{\mathscr{E}^4} h^{\lambda\rho}\nabla_{\sigma}\mathscr{E}\cdot\nabla_{\rho}\langle P^{\sigma}\rangle, \\ \hat{c}^{\lambda}_{(6)} &= \frac{\mathscr{K}^{5/2}}{\mathscr{E}^3} h^{\lambda\rho}\nabla_{\sigma}\mathscr{E}\cdot\nabla_{\rho}\langle u^{\sigma}\rangle, \quad \hat{c}^{\lambda}_{(7)} = \frac{\mathscr{K}^{5/2}}{\mathscr{E}^3} h^{\rho\sigma}\nabla_{\rho}\mathscr{E}\cdot\nabla_{\sigma}\langle u^{\lambda}\rangle. \end{aligned}\right\} \tag{5.24}$$

Again, in both expansions the restriction of being a space-like diffusion vector excluded terms that are proportional to uncontracted mean 4-velocities $\langle u^{\alpha}\rangle$. However, the mean 4-velocities enter here in a contracted form giving rise to a natural inclusion of material time derivatives. Since these terms $b^{\lambda}_{(3)} = \hat{c}^{\lambda}_{(3)}$ even represent the kinematic left-hand sides of the mean averaged Navier–Stokes momentum equations (5.12), they can explicitly account for turbulent memory effects during any process of turbulent diffusion.

Expanding the first high-Re_T dissipation production term $\hat{\mathscr{P}}^{\alpha}_{(1)\beta}(\mathscr{V}_{\tau})$ (5.17) up to the desired linear order, with its restrictions mentioned in § 5, will result in

$$\hat{\mathscr{P}}^{\alpha}_{(1)\beta} = \mathscr{E} \sum_{i=1}^{6} \hat{\Xi}_{(i)} \hat{d}^{\alpha}_{\beta(i)}, \tag{5.25}$$

with $\hat{\Xi}_{(i)} = \hat{\Xi}_{(i)}(\mathscr{I}_{(j)})$ and the spatial–dimensionless expansion terms as

$$\left.\begin{aligned} &\hat{d}^{\alpha}_{\beta(1)} = h^{\alpha\lambda}k^{\langle u\rangle}_{\lambda\beta}, \\ \hat{d}^{\alpha}_{\beta(2)} = \frac{\mathscr{K}^2}{\mathscr{E}^2}\nabla_\beta\langle P^\alpha\rangle,\ \ &\hat{d}^{\alpha}_{\beta(3)} = \frac{\mathscr{K}^2}{\mathscr{E}^2}h^{\alpha\rho}k^{\langle u\rangle}_{\sigma\beta}\nabla_\rho\langle P^\sigma\rangle,\ \ \hat{d}^{\alpha}_{\beta(4)} = \frac{\mathscr{K}^2}{\mathscr{E}^2}h^{\alpha\rho}k^{\langle u\rangle}_{\rho\beta}\nabla_\sigma\langle P^\sigma\rangle, \\ &\hat{d}^{\alpha}_{\beta(5)} = \frac{\mathscr{K}}{\mathscr{E}}\nabla_\beta\langle u^\alpha\rangle,\ \ \hat{d}^{\alpha}_{\beta(6)} = \frac{\mathscr{K}}{\mathscr{E}}h^{\alpha\rho}k^{\langle u\rangle}_{\sigma\beta}\nabla_\rho\langle u^\sigma\rangle. \end{aligned}\right\} \quad (5.26)$$

The zeroth-order term $\hat{d}^{\alpha}_{\beta(1)}$ as well as the linear term $\hat{d}^{\alpha}_{\beta(4)}$ will give no contributions in the $\mathscr{E}$-transport equation (5.12), due to the incompressibility constraint of the mean velocity field. This holds in 'all' Newtonian reference frames, since if a tensorial object vanishes in one reference frame it vanishes in all reference frames, the major feature of the tensor concept. Furthermore, since the time-like component $\beta = 0$ gives no contribution in the $\mathscr{E}$-equation and since the two metrical tensors $h^{\alpha\beta}$ and $k^{\langle u\rangle}_{\alpha\beta}$ are inverse tensors in every spatial subspace, the two terms $\hat{d}^{\alpha}_{\beta(2)}$ and $\hat{d}^{\alpha}_{\beta(3)}$ always turn out to be equal due to the commutativity of the covariant derivative operator. Thus, relative to the $\mathscr{E}$-transport equation, the above linear expansion actually consists of only three terms.

A linear expansion of the second high-Re_T dissipation production term $\hat{\mathscr{P}}^{\alpha\beta}_{(2)\lambda}(\mathscr{V}_\tau)$ (5.17) together with its corresponding restrictions leads to an empty expansion, an expansion with no terms involved. A polynomial expansion of $\hat{\mathscr{P}}^{\alpha\beta}_{(2)\lambda}(\mathscr{V}_\tau)$ will always start off quadratically. The only possible linear term, proportional to $h^{\alpha\beta}k^{\langle u\rangle}_{\lambda\sigma}\langle u^\sigma\rangle$, vanishes in all Newtonian reference frames, since $k^{\langle u\rangle}_{\lambda\sigma}\langle u^\sigma\rangle = 0$ for all λ. That the production term $\hat{\mathscr{P}}^{\alpha\beta}_{(2)\lambda}(\mathscr{V}_\tau)$ really acts as a higher-order contribution can be seen when using the direct simulation data of channel flows as a reference (Mansour, Kim & Moin 1988). The term proportional to $\hat{\mathscr{P}}^{\alpha\beta}_{(2)\lambda}$ is negligible small when compared to all other $\mathscr{E}$-contributions in the regime of high Re_T (Rodi & Mansour 1993). Certainly the channel flow is no rigorous reference when investigating more complex flow configurations. But nevertheless, at this level of development the four-dimensional formulation generally demonstrates a higher-order behaviour for $\hat{\mathscr{P}}^{\alpha\beta}_{(2)\lambda}$.

The last unclosed term to be modelled is the high-Re_T dissipation scalar $\hat{\Psi}(\mathscr{V}_\tau)$. Up to quadratic order this scalar is just a function of invariants (5.18) as is the case for all previously given expansion coefficients,

$$\hat{\Psi} = \frac{\mathscr{E}^2}{\mathscr{K}}\hat{f}(\mathscr{I}_j), \quad 1 \leqslant j \leqslant 12. \quad (5.27)$$

Finally to close this section, it is worthwhile to note that since we have approached the concept of turbulence modelling by modelling a nonlinear EVM in its most systematic and general way, it is not surprising to see that the model proposed herein will reduce to numerous standard EVMs when choosing the expansion coefficients for the unclosed terms appropriately. For example the choice

$$\left.\begin{aligned} &\Lambda_{(2)} = -C_\mu,\ \ \Lambda_{(i)} = 0,\ i \neq 2, \\ \Theta_{(1)} = \frac{C_\mu}{\sigma_k},\ \ \Theta_{(i)} = 0,\ i \neq 1,\ \ \ &\hat{\Phi}_{(1)} = \frac{C_\mu}{\sigma_\epsilon},\ \ \hat{\Phi}_{(i)} = 0,\ i \neq 1, \\ \hat{\Xi}_{(5)} = \hat{\Xi}_{(6)} = C_\mu C_{\epsilon 1},\ \ &\hat{\Xi}_{(i)} = 0,\ i \neq 5, 6,\ \ \ \hat{f} = C_{\epsilon 2}, \end{aligned}\right\} \quad (5.28)$$

with the constant numerical values

$$C_\mu = 0.09,\ \sigma_k = 1.0,\ \sigma_\epsilon = 1.3,\ C_{\epsilon 1} = 1.44,\ C_{\epsilon 2} = 1.92, \tag{5.29}$$

will reduce the invariant high-Re_T model into the standard linear k–ϵ model of Jones & Launder (1972) with the model coefficients suggested by Launder & Sharma (1974). If we choose the expansion coefficients for the Reynolds-stress tensor for example as

$$\Lambda_{(2)} = -C_\mu,\ \ \Lambda_{(7)} = -\Lambda_{(8)} = C_2,\ \ \Lambda_{(i)} = 0,\ i \neq 2, 7, 8, \tag{5.30}$$

$$\text{with } C_\mu = \frac{1}{6.5 + A_s^* \bar{U}^*},\quad C_2 = \frac{\sqrt{1 - 9C_\mu^2 \bar{S}^{*2}}}{1 + 6\bar{S}^* \bar{\Omega}^*},$$

where $\frac{1}{2}\sqrt{6} \leqslant A_s^* \leqslant \sqrt{6}$, $\bar{U}^* = \sqrt{\mathscr{I}_{11}}$, $\bar{S}^* = \sqrt{\frac{1}{2}(\mathscr{I}_{11} + \mathscr{I}_{12})}$, $\bar{\Omega}^* = \sqrt{\frac{1}{2}(\mathscr{I}_{11} - \mathscr{I}_{12})}$,

and the remaining coefficients for the $\mathscr{K}$- and $\mathscr{E}$-equation as before, we end up with the Reynolds stress algebraic model of Shih *et al.* (1995) – in their full model the quantity A_s^* is determined dynamically by an algebraic equation containing a cubic invariant, which of course cannot be presented anymore by the invariant high-Re_T model due to its quadratic nonlinearity.

More interesting, however, is to demonstrate whether this systematic four-dimensional formulation is able to *a priori* rule out certain model constructions. In other words, can this four-dimensional EVM possibly exclude already existing EVMs or other models that have been successfully presented in the literature so far? For that, one really has to recall the two obvious physical differences between the true four-dimensional invariant EVM proposed herein and those EVMs which have been presented so far: (i) the four-dimensional invariant model 'systematically' distinguishes between space-like and time-like closures, which, up to its degree in nonlinearity, assures the correct treatment of curvature and memory effects in the flow respectively, and (ii) with the independent aid of a Lie-group symmetry analysis the mean pressure gradient is 'systematically' introduced as an own closure variable, which, in the form of the mean pressure Hessian, shows the ability to model non-local flow effects. Now, if these characteristic features are all set to zero, one certainly is left with those standard EVMs as shown exemplary above. On the other hand, if the mean pressure and/or any arbitrary time derivatives are introduced as closure elements in a very *ad hoc* way for building an invariant turbulence model without having the appropriate mathematical framework at hand, it is very unlikely that such a model will be supported by a corresponding construction within the four-dimensional formulation. For example, when only regarding the concept of time derivatives, this is the case for the nonlinear K–l and K–ϵ models proposed by Speziale (1987). Therein the Euclidean frame-indifferent Oldroyd time derivative has been heuristically introduced as a closure variable for the Reynolds-stress tensor within the usual (3+1)-dimensional geometrical setting. The result is that absolutely no choice can be made for the expansion coefficients in the corresponding four-dimensional invariant EVM such that it allows for a reduction to those quadratic models developed in Speziale (1987). Hence these set of models are 'not' included within the invariant four-dimensional formulation. This example finally demonstrates the relevance and importance of using the appropriate mathematical framework when aiming at modelling turbulent flows.

Thus, altogether, the herein-proposed four-dimensional invariant high-Re_T model can be seen as a promising extension of current nonlinear EVMs, not only to have at the end a fully universal discussion on curvature effects but also to have a fully universal discussion on non-stationary effects, in particular on memory effects within any turbulent flow.

6. Discussion

The aim of this paper was not to offer a ready-to-simulate model but rather to provide a new and especially a natural mathematical framework in which invariant turbulence modelling could be performed. The advantages of a four-dimensional modelling approach in all its facets was demonstrated at the example of constructing a new nonlinear EVM within the k–ϵ family. The question of how far this model is really quantitatively superior over existing nonlinear EVMs has to await further model development.

Surely, the invariant high-Re_T model as proposed in the previous section still needs to be reduced by making use of physical modelling constraints like rapid distortion theory, realizability and most importantly a Lie-group symmetry analysis based on all invariant 'solutions', in particular on all scaling laws of the Navier–Stokes equations given by the corresponding optimal system of subalgebras (Ovsiannikov 1982; Fushchych & Popowych 1994*a*,*b*; Andreev *et al.* 1998), which will be subject to future research. Finally it needs to be calibrated by means of experimental and simulated data from basic flow configurations.

It should be clear that the model itself is not universally superior, since it is based on various assumptions in the choice of the tensor dependency set $\mathscr{V}$ and its subsequent polynomial expansion, but that the mathematical framework, the four-dimensional formulation, is superior over the (3+1)-dimensional formulation, which certainly is independent of any choice made on $\mathscr{V}$. This should encourage the use of this formulation also for modelling Reynolds-stress transport equations, or for closing two-point correlation equations, or in subgrid-scale modelling of LES.

A true four-dimensional modelling approach, which automatically generates additional restrictions which again do not exist in the usual (3+1)-dimensional framework, guarantees a universal structure of the proposed models in 'all' Newtonian reference frames, without exceptions. Hence, in a four-dimensional framework it is not necessary to model separately the inertial case from the non-inertial case. A certain model chosen will describe equally well or equally bad the non-inertial case as it would describe the inertial counterpart, irrespective of if the inertial forces are induced by time-dependent rotations or by the more complicated phenomenon of curved surfaces.

A closure novelty, besides the four-dimensional aspects, was to include next to the mean velocity gradient the mean pressure Hessian and the mean velocity field itself. The idea for an improved local algebraic model with minimum mathematical complexity was to make use of all internal flow information the averaged Navier–Stokes equations could offer and not, except the two turbulent scales of length and time, to pull in additional turbulent information from outside, as the algebraic structure-based models of Reynolds & Kassinos (1995) do, which results in additional equations for turbulence-structure parameters – the aim of these models is to sensitize the Reynolds stresses to coherent structures in a turbulent flow but at the expense of mathematical simplicity.

Including the mean pressure as a new internal modelling variable, we could specifically show that the phenomenon of swirl in axially rotating turbulent pipe flows can already be captured by a quadratic nonlinearity, and at the same time giving a more advanced interpretation in that swirl also has, besides the mean velocity, its origin in the coupling of mean pressure and velocity gradients. Furthermore, the pressure inclusion may generally also account for non-local effects in mean deformations, a topic still not captured at all by current nonlinear EVMs.

A minor drawback when including the pressure is that numerics becomes slightly more challenging. One has a strong coupled system between mean velocity and mean pressure. The mean pressure can no longer be decoupled via the Poisson equation anymore as it can be done for current turbulence models. It rather has to be treated, next to the three mean velocity components, as an additional fourth flow component within the numerical scheme used.

A further novelty was to apply the same closure strategy used for the Reynolds-stress tensor also to all unclosed terms in the $\mathscr{K}$- and $\mathscr{E}$-equation. There we could see how the velocity field in each equation gave rise to a natural dependence on a material time derivative. Hence the modelled equations are able to explicitly account for turbulent memory effects within any Newtonian reference frame.

In all, this paper reveals that modelling on a four-dimensional manifold leads to qualitatively new and more profound modelling restrictions which the usual (3+1)-dimensional framework is unable to provide. Regarding turbulence it offers a 'unified' invariant turbulence modelling approach.

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Appendix A. Compendium on tensor analysis

For the purpose of consistent notation and easy reference the most important concepts and relations of tensor analysis are compiled. Further details and derivations can be taken for example from the excellent book of Schrödinger (1985).

Differential geometry formulates a clear distinction between a manifold and a coordinate system. A manifold of dimension N is to be seen as a set or a space of points possible to embed coordinate systems which locally assign to every point of the manifold a unique N-tuple of real values, the coordinates of the manifold. The manifold itself is defined as essential and thus immutable, whereas the embedding of a coordinate system into the manifold is not unique; it can be chosen freely by transforming the coordinates accordingly – the choice of a coordinate system is a matter of expediency, not of truth.

Without loss of generality let us focus on a four-dimensional continuous and differentiable manifold $\mathscr{M}$ whose points are distinguished from each other by assigning four real values x^0, x^1, x^2, x^3 to each of them. However, this first labelling should have no prerogative over any other one,

$$\left.\begin{aligned} \tilde{x}^0 &= \tilde{x}^0(x^0,\ldots,x^3), \quad & \tilde{x}^1 &= \tilde{x}^1(x^0,\ldots,x^3),\\ \tilde{x}^2 &= \tilde{x}^2(x^0,\ldots,x^3), \quad & \tilde{x}^3 &= \tilde{x}^3(x^0,\ldots,x^3), \end{aligned}\right\} \tag{A 1}$$

where $\tilde{x}^\alpha$ are four continuous and differentiable functions of x^α, such that their functional determinant vanishes nowhere. This is necessary in order to secure a one-to-one correspondence between the two sets of labels. But sometimes local exceptions have to be made, for example the point of origin which in the transition from a Cartesian to a polar coordinate system is to be excluded.

Now, tensor analysis aims at looking for mathematical entities, numbers or sets of numbers to which a meaning can be attached to every point $\boldsymbol{x}$ in a given manifold

$\mathscr{M}$ regarding arbitrary coordinate transformations: if we consider a coordinate transformation as given in (A 1), a contravariant vector A^α is defined as a quantity with four components which transform like the coordinate differentials $\mathrm{d}x^\alpha$ in (A 1), which means

$$\tilde{A}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} A^\beta. \tag{A 2}$$

Thus the coordinate differentials themselves form a contravariant vector, but for the finite coordinate differences Δx^α this is only the case if transformation (A 1) is linear and for the coordinates x^α themselves only if it is also homogeneous. A quantity B_α is called a covariant vector if its components transform as

$$\tilde{B}_\alpha = \frac{\partial x^\beta}{\partial \tilde{x}^\alpha} B_\beta. \tag{A 3}$$

In general one cannot unambiguously associate a covariant and a contravariant vector with each other; for this one needs more information on the inner structure of the manifold. One can now define covariant, contravariant and mixed tensors of any rank by similar expressions,

$$\tilde{T}^{\alpha\beta\dots}_{\mu\nu\dots} = \frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}\frac{\partial \tilde{x}^\beta}{\partial x^\lambda}\cdots\frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\cdots T^{\kappa\lambda\dots}_{\rho\sigma\dots}, \tag{A 4}$$

where a vector is a tensor of rank 1 and a scalar an invariant tensor of rank 0. Note that if a coordinate transformation carries a special name like orthogonal, Lorentz, Galilei or Euclidean, all tensors within such a transformation will adopt this name in calling themselves orthogonal, Lorentz, Galilei or Euclidean tensors respectively. Orthogonal tensors are also called isotropic tensors.

A tensor is thus solely defined by its transformation property in that it has to transform linearly and homogeneously. This property guarantees that if a tensor is zero in one coordinate system it is zero in all coordinate systems. Without loosing the tensor character in (A 4) one can define an autonomous tensor algebra for each point $\boldsymbol{x}$ in the manifold $\mathscr{M}$, which means that referring only to a single point in the manifold one can define addition and subtraction for tensors of the same rank and multiplication and contraction for tensors of any rank.

When defining ordinary differentiation for tensors, however, the tensor character of (A 4) in general will be lost: partial derivatives of tensors show no tensor transformation behaviour, since they emerge from subtraction of tensors which do not refer to a single point but which refer to different points of the manifold. That ordinary differentiation in general breaks the tensor character can be easily seen for example by differentiating the transformation rule (A 3) of a covariant vector,

$$\frac{\partial \tilde{B}_\alpha}{\partial \tilde{x}^\mu} = \frac{\partial x^\beta}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\frac{\partial B_\beta}{\partial x^\nu} + \frac{\partial^2 x^\beta}{\partial \tilde{x}^\mu \partial \tilde{x}^\alpha} B_\beta, \tag{A 5}$$

where the second inhomogeneous term, except for linear coordinate transformations, prevents the derivative of a covariant vector to be a tensor of rank $(0, 2)$.

Nevertheless, to define differentiation which maintains the tensor character of any expression in (A 4) one has to either specify the inner structure of the manifold or be in the possession of a contravariant vector field already defined in the manifold, since during differentiation different points of the manifold need to be connected. The former leads us to affinely and metrically connected manifolds, while the latter leads us to the notion of the Lie derivative.

Let V^λ be a contravariant vector field in the manifold $\mathscr{M}$. The 'Lie derivative' of a tensor $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ along the vector field V^λ is a tensor of the same rank as $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ defined by

$$\mathscr{L}_V T^{\alpha\beta\ldots}_{\mu\nu\ldots} := V^\lambda \partial_\lambda T^{\alpha\beta\ldots}_{\mu\nu\ldots} - T^{\lambda\beta\ldots}_{\mu\nu\ldots}\partial_\lambda V^\alpha - T^{\alpha\lambda\ldots}_{\mu\nu\ldots}\partial_\lambda V^\beta - \cdots + T^{\alpha\beta\ldots}_{\lambda\nu\ldots}\partial_\mu V^\lambda + T^{\alpha\beta\ldots}_{\mu\lambda\ldots}\partial_\nu V^\lambda + \cdots, \tag{A 6}$$

with a negative term for each contravariant index and a positive term for each covariant index. Preserving the rank of the tensor the Lie derivative $\mathscr{L}_V$ satisfies all rules of ordinary differentiation ∂_ν as the product rule and linearity. Roughly speaking one can say that the Lie derivative is the directional derivative of a tensor along a curve set by a vector field adjusted for the change in the tangent: to differentiate the tensor $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ at a point P one drags the tensor along the curve given by the vector field V^λ through P to a neighbouring point Q; the derivative then compares the dragged tensor with the tensor evaluated at Q in the limit $Q \to P$.

A.1. *Affine and metric spaces*

To define an arbitrary differentiation in a manifold $\mathscr{M}$ which maintains tensor character, one has to introduce an extra geometric structure on the manifold, an 'affine connection' $\Gamma^\sigma_{\kappa\lambda}$ with the transformation law

$$\tilde{\Gamma}^\rho_{\mu\nu} = \frac{\partial\tilde{x}^\rho}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial\tilde{x}^\mu}\frac{\partial x^\lambda}{\partial\tilde{x}^\nu}\Gamma^\sigma_{\kappa\lambda} + \frac{\partial\tilde{x}^\rho}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}. \tag{A 7}$$

Thus $\Gamma^\sigma_{\kappa\lambda}$ does not transform as a tensor of rank $(1, 2)$ except under linear coordinate transformations: an affine connection which vanishes in one coordinate system, $\Gamma^\sigma_{\kappa\lambda} = 0$, does not necessarily vanish in any other coordinate system, $\tilde{\Gamma}^\rho_{\mu\nu} \neq 0$. The affine connection itself can be arbitrarily assigned in one coordinate system and determines the meaning of parallel displacement in the space considered. Then one can define covariant derivatives by

$$\nabla_\nu T^{\kappa\lambda\ldots}_{\rho\sigma\ldots} := \partial_\nu T^{\kappa\lambda\ldots}_{\rho\sigma\ldots} + \Gamma^\kappa_{\alpha\nu}T^{\alpha\lambda\ldots}_{\rho\sigma\ldots} + \Gamma^\lambda_{\alpha\nu}T^{\kappa\alpha\ldots}_{\rho\sigma\ldots} + \cdots - \Gamma^\alpha_{\rho\nu}T^{\kappa\lambda\ldots}_{\alpha\sigma\ldots} - \Gamma^\alpha_{\sigma\nu}T^{\kappa\lambda\ldots}_{\rho\alpha\ldots} - \cdots, \tag{A 8}$$

where the covariant derivative operator ∇_ν satisfies all rules of ordinary differentiation ∂_ν. If $T^{\kappa\lambda\ldots}_{\rho\sigma\ldots}$ is a tensor of rank (r, s), then $\nabla_\nu T^{\kappa\lambda\ldots}_{\rho\sigma\ldots}$ is a tensor of $(r, s + 1)$. In the following we are only concerned with symmetric connections $\Gamma^\sigma_{\kappa\lambda} = \Gamma^\sigma_{\lambda\kappa}$, that is with affine connected manifolds which are free of torsion.

If we form the second covariant derivatives by successive applications of (A 8), then in general the differentiations with respect to different coordinates do not commute. This property of the affine space is called curvature and is described by the Riemann curvature tensor

$$R^\kappa_{\mu\lambda\nu} = \partial_\lambda\Gamma^\kappa_{\mu\nu} - \partial_\nu\Gamma^\kappa_{\mu\lambda} + \Gamma^\kappa_{\rho\lambda}\Gamma^\rho_{\mu\nu} - \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\mu\lambda}. \tag{A 9}$$

Since the Riemann curvature is a tensor it will be zero in all coordinate systems if it is zero in one specified coordinate system.

Up to now we have not discussed how to measure length in a manifold. In an affine space the concept of length can only be defined along a geodesic, but lengths along different geodesics cannot be compared. (A curve in an affine space is called a geodesic if its development is a straight line; in other words geodesics in affine spaces are defined to be curves whose tangent vectors remain parallel if they are transported along them.) Such a comparison becomes possible if the space admits a symmetric tensor of rank 2 with vanishing covariant derivatives, the metric tensor $g_{\mu\nu}$ – the assumption that the covariant derivative of the metric tensor should vanish guarantees the natural demand that the length of vectors does not change under a parallel transport. Then the distance ds in a manifold is determined by

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \tag{A 10}$$

which by suitable parameterization agrees with the distance along geodesics.

If the determinant of the $g_{\mu\nu}$ vanishes, the metric is called singular; if it does not vanish anywhere, the space is called metric or Riemannian. In such a space the geodesics are constructed by a variation principle as the shortest as well as the straightest lines. The assumption $\nabla_\rho g_{\mu\nu} = 0$ give rise to equations, which can be solved for the affine connections,

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}), \tag{A 11}$$

where the affine connections now carry the name of 'Christoffel symbols'; the tensor $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ defined by

$$g_{\mu\rho} g^{\rho\nu} = \delta^\nu_\mu. \tag{A 12}$$

With the help of these two tensors one now can uniquely associate with each contravariant vector A^ρ a covariant vector $A_\mu = g_{\mu\rho}A^\rho$, and conversely $A^\rho = g^{\rho\mu}A_\mu$.

Any symmetric tensor of rank 2 such as $g_{\mu\nu}$ which can be associated with a quadratic form can be characterized by its signature, that is by the difference between the number of positive and negative terms after the quadratic form has been diagonalized. This number is an invariant by Sylvester's law of inertia of quadratic forms.

Appendix B. The $\tau^{r\varphi}$-component in axially rotating pipe flows

Following the reasoning of Speziale *et al.* (2000) the off-diagonal shear component of the Reynolds-stress tensor $\tau^{r\varphi}$ is the source for generating a non-zero mean swirl velocity $\langle u_\varphi\rangle$ in an axially rotating pipe. Their result, that it arises from a cubic nonlinearity in conventional algebraic stress models through the constant presence of the mean axial velocity, is extended herein with the insight that already a quadratic nonlinearity will suffice to capture this secondary flow effect if only the Reynolds-stress tensor is modelled as presented in (5.19). In this case not the mean axial velocity $\langle u_z\rangle$, if the pipes symmetry axis is aligned with the z-axis, but the mean radial pressure gradient $\partial_r\langle p\rangle$ will maintain the swirl.

To demonstrate it we identically proceed as in Speziale *et al.* (2000): we initially focus on a fully developed turbulent non-rotating pipe flow on to which an axial system rotation is then superposed. The fully developed non-rotating pipe flow is a one-dimensional, one-component flow configuration, characterized by the radial-dependent mean axial velocity profile $\langle u_z\rangle(r)$, which again is driven by a mean pressure $\langle p\rangle(r,z)$ with a constant gradient in the axial direction $\partial_z^2\langle p\rangle = 0$. Now, if the axial rotation is switched on, these are the only mean flow variables which can account for a non-zero turbulent shear stress $\tau^{r\varphi}$. To describe this flow it is reasonable to change to the co-rotating frame. Relative to the standard reference frame (Cartesian and inertial) the transformation to axial co-rotating cylindrical coordinates is simply given as

$$\left.\begin{aligned} x &= r\cos(\varphi+\omega), \\ y &= r\sin(\varphi+\omega), \\ z &= z, \end{aligned}\right\} \quad \xleftrightarrow{\text{Inverse}} \quad \left\{\begin{aligned} r &= \sqrt{x^2+y^2}, \\ \varphi &= \arctan(y/x) - \omega, \\ z &= z, \end{aligned}\right.$$

where $\omega = \omega(t)$ is the time-dependent azimuthal angle induced by the axial rotation. If we identify

$$x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z \quad \text{and} \quad \tilde{x}^0 = t,\ \tilde{x}^1 = r,\ \tilde{x}^2 = \varphi,\ \tilde{x}^3 = z \tag{B 1}$$

and the mean 4-velocities $\langle u^\alpha \rangle$, $\langle \tilde{u}^\alpha \rangle$ accordingly, the two space-like metrical tensors (4.5), which transform as

$$\tilde{h}^{\alpha\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^\rho}\frac{\partial \tilde{x}^\beta}{\partial x^\sigma} h^{\rho\sigma}, \quad \tilde{k}^{\langle \tilde{u} \rangle}_{\alpha\beta} = \frac{\partial x^\rho}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\beta} k^{\langle u \rangle}_{\rho\sigma}, \tag{B 2}$$

will attain the following spatial–diagonal structure in a co-rotating cylindrical frame:

$$\tilde{h}^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{k}^{\langle \tilde{u} \rangle}_{\alpha\beta} = \begin{pmatrix} \|\langle \tilde{\boldsymbol{u}} \rangle\|^2 & -\langle u_r \rangle & -r^2\langle u_\varphi \rangle & -\langle u_z \rangle \\ -\langle u_r \rangle & 1 & 0 & 0 \\ -r^2\langle u_\varphi \rangle & 0 & r^2 & 0 \\ -\langle u_z \rangle & 0 & 0 & 1 \end{pmatrix}, \tag{B 3}$$

with $\|\langle \tilde{\boldsymbol{u}} \rangle\|^2 = \tilde{k}^{\langle \tilde{u} \rangle}_{ij} \langle \tilde{u}^i \rangle \langle \tilde{u}^j \rangle = \langle u_r \rangle^2 + r^2 \langle u_\varphi \rangle^2 + \langle u_z \rangle^2$. Since the affine connection $\Gamma^\lambda_{\alpha\beta}$ vanishes in the standard reference frame, the determination of its components according to rule (A 7) for the above coordinate transformation simply reduces to

$$\tilde{\Gamma}^\lambda_{\alpha\beta} = \frac{\partial \tilde{x}^\lambda}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}. \tag{B 4}$$

Of its 40 components only 4 are non-zero:

$$\Gamma^r_{0\varphi} = -r\dot{\omega}, \quad \Gamma^\varphi_{0r} = \frac{\dot{\omega}}{r}, \quad \Gamma^r_{\varphi\varphi} = -r, \quad \Gamma^\varphi_{r\varphi} = \frac{1}{r}. \tag{B 5}$$

Now, the transformed Reynolds-stress tensor $\tilde{\tau}^{\alpha\beta}$ of the high-Re_T turbulence model proposed in § 5.1 has, by construction, the same form as in (5.19) and (5.20). When determining the relevant transformed shear component $\tau^{r\varphi}$ it shows that only the expansion term proportional to $a^{r\varphi}_{(5)}$ gives a contribution, which we now want to write out explicitly, having in mind that the covariant derivative of a scalar quantity, as the mean pressure, is just its partial derivative $\nabla_\alpha \langle p \rangle = \partial_\alpha \langle p \rangle$, that $\tilde{h}^{\alpha\beta}$ is diagonal and that we consider a fully developed flow situation before rotation is switched on, which means $\langle u_r \rangle = \langle u_\varphi \rangle = 0$, $\partial_\varphi \langle u_z \rangle = \partial_z \langle u_z \rangle = 0$ and $\partial_\varphi \langle p \rangle = 0$, $\partial_z^2 \langle p \rangle = 0$:

$$\left.\begin{aligned}
\tau^{r\varphi} \sim a^{r\varphi}_{(5)} &\sim (\tilde{h}^{r\rho}\tilde{h}^{\varphi\sigma} + \tilde{h}^{\varphi\rho}\tilde{h}^{r\sigma})\tilde{\nabla}_\rho \tilde{\nabla}_\lambda \langle \tilde{p} \rangle \cdot \tilde{\nabla}_\sigma \langle \tilde{u}^\lambda \rangle \\
&= h^{rr}h^{\varphi\varphi}\nabla_r \tilde{\nabla}_\lambda \langle p \rangle \cdot \nabla_\varphi \langle \tilde{u}^\lambda \rangle + h^{rr}h^{\varphi\varphi}\nabla_\varphi \tilde{\nabla}_\lambda \langle p \rangle \cdot \nabla_r \langle \tilde{u}^\lambda \rangle \\
&= \tfrac{1}{r^2}\nabla_r(\tilde{\partial}_\lambda \langle p \rangle) \cdot \nabla_\varphi \langle \tilde{u}^\lambda \rangle + \tfrac{1}{r^2}\nabla_\varphi(\tilde{\partial}_\lambda \langle p \rangle) \cdot \nabla_r \langle \tilde{u}^\lambda \rangle \\
&= \tfrac{1}{r^2}\left[\partial_r \tilde{\partial}_\lambda \langle p \rangle - (\tilde{\partial}_\rho \langle p \rangle)\tilde{\Gamma}^\rho_{r\lambda}\right] \cdot \left[\partial_\varphi \langle \tilde{u}^\lambda \rangle + \langle \tilde{u}^\sigma \rangle \tilde{\Gamma}^\lambda_{\varphi\sigma}\right] \\
&\quad + \tfrac{1}{r^2}\left[\partial_\varphi \tilde{\partial}_\lambda \langle p \rangle - (\tilde{\partial}_\rho \langle p \rangle)\tilde{\Gamma}^\rho_{\varphi\lambda}\right] \cdot \left[\partial_r \langle \tilde{u}^\lambda \rangle + \langle \tilde{u}^\sigma \rangle \tilde{\Gamma}^\lambda_{r\sigma}\right] \\
&= \tfrac{1}{r^2}\partial_r^2 \langle p \rangle \cdot \langle \tilde{u}^\sigma \rangle \tilde{\Gamma}^r_{\varphi\sigma} - \tfrac{1}{r^2}(\partial_r \langle p \rangle)\tilde{\Gamma}^r_{\varphi\lambda} \cdot \left[\partial_r \langle \tilde{u}^\lambda \rangle + \langle \tilde{u}^\sigma \rangle \tilde{\Gamma}^\lambda_{r\sigma}\right] \\
&= -\tfrac{\dot{\omega}}{r}\partial_r^2 \langle p \rangle + \tfrac{\dot{\omega}}{r^2}\partial_r \langle p \rangle.
\end{aligned}\right\} \tag{B 6}$$

Thus a mean swirl velocity $\langle u_\varphi \rangle$ will be generated in an axially rotating pipe and persists in a fully developed flow, since the radial pressure gradient $\partial_r \langle p \rangle$ gives rise to a non-zero Reynolds shear stress $\tau^{r\varphi}$, independent of the swirl velocity. If the rotation ceases, $\dot{\omega} = 0$, the mean swirl velocity will decay, ending in a fully developed non-rotating pipe flow, which exhibits only a mean axial velocity component $\langle u_z \rangle$.

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